pySecDec Documentation

Release 1.4.4

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Contents

pySecDec [PSD17], [PSD18] is a toolbox for the calculation of dimensionally regulated parameter integrals using the sector decomposition approach [BH00]; see also [Hei08], [BHJ+15].

Please cite the following references if you use *pySecDec* for a scientific publication:

- pySecDec [PSD17], [PSD18]
- CUBA [Hah05], [Hah16]
- FORM [Ver00], [KUV13], [RUV17]
- GSL [GSL]
- nauty [MP+14] (if you use *dreadnaut*)
- normaliz [BIR], [BIS16] (if you use a geometric decomposition strategy)
- QMC [LWY+15] (if you use the quasi-monte carlo integrator)

CHAPTER 1

Installation

1.1 Download the Program and Install

pySecDec should run fine with both, *python* 2.7 and *python* 3 on unix-like systems.

Before you install *pySecDec*, make sure that you have recent versions of *numpy* (http://www.numpy.org/) and *sympy* (http://www.sympy.org/) installed. The version of *sympy* should be 0.7.6 or higher, the version of *numpy* should be 1.6 or higher. Type

```
$ python -c "import numpy"
$ python -c "import sympy"
```

to check for their availability.

In case either *numpy* or *sympy* are missing on your machine, it is easiest to install them from your package repository. Alternatively, and in particular if you do not have administrator rights, *pip* (https://pip.pypa.io/en/stable/) may be used to perform the installation.

To install *pySecDec* download and upack the tarball from http://secdec.hepforge.org/. The tarball contains a distribution of *pySecDec* and the additional dependencies listed *below*. Typing

```
$ make
```

should build all redistributed packages and display two commands to be added to your `.bashrc` or `.profile`.

1.2 The Geomethod and Normaliz

Note: If you are not urgently interested in using the `geometric decomposition`, you can ignore this section for the beginning. The instructions below are not essential for a *pySecDec* installation. You can still install *normaliz* **after** installing *pySecDec*. All but the `geometric decomposition` routines work without *normaliz*.

If you want to use the *geometric decomposition* module, you need the *normaliz* [BIR] command line executable. The *geometric decomposition* module is designed for *normaliz* version 3 - currently versions 3.3.0, 3.4.0, 3.5.4, 3.6.0, 3.6.2, 3.7.3, 3.7.4, and 3.8.1 are known to work. We recommend to set your $PATH such that the *normaliz* executable is found. Alternatively, you can pass the path to the *normaliz* executable directly to the functions that need it.

1.3 Drawing Feynman Diagrams with *neato*

In order to use `plot_diagram()`, the command line tool *neato* must be available. The function `loop_package()` tries to call `plot_diagram()` if given a `LoopIntegralFromGraph` and issues a warning on failure. That warning can be safely ignored if you are not interested in the drawing.

neato is part of the *graphviz* package. It is available in many package repositories and at http://www.graphviz.org.

1.4 Additional Dependencies for Generated c++ Packages

The intended main usage of *pySecDec* is to make it write c++ packages using the functions `pySecDec.code_writer.make_package()` and `pySecDec.loop_integral.loop_package()`. In order to build these c++ packages, the following additional non-python-based libraries and programs are required:

- CUBA (http://www.feynarts.de/cuba/)
- QMC (https://github.com/mppmu/qmc)
- FORM (http://www.nikhef.nl/~form/)
- SecDecUtil (part of *pySecDec*, see *SedDecUtil*), depends on:
  - catch (https://github.com/philsquared/Catch)
  - gsl (http://www.gnu.org/software/gsl/)

The functions `pySecDec.code_writer.make_package()` and `pySecDec.loop_integral.loop_package()` can use the external program *nauty* [MP+14] to find all sector symmetries and therefore reduce the number of sectors:

- NAUTY (http://pallini.di.uniroma1.it/)

These packages are redistributed with the *pySecDec* tarball; i.e. you don't have to install any of them yourself.

CHAPTER 2

Getting Started

After installation, you should have a folder *examples* in your main *pySecDec* directory. Here we describe a few of the examples available in the *examples* directory. A full list of examples is given in *List of Examples*.

2.1 A Simple Example

We first show how to compute a simple dimensionally regulated integral:

$$\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \; (x+y)^{-2+\epsilon}.$$

To run the example change to the *easy* directory and run the commands:

```
$ python generate_easy.py
$ make -C easy
$ python integrate_easy.py
```

Additional build options are discussed in the *next section*. This will evaluate and print the result of the integral:

```
Numerical Result: + (1.00015897181235158e+00 +/- 4.03392522752491021e-03)*eps^-1 + (3.
↪06903035514056399e-01 +/- 2.82319349818329918e-03) + O(eps)
Analytic Result: + (1.000000)*eps^-1 + (0.306853) + O(eps)
```

The file `generate_easy.py` defines the integral and calls *pySecDec* to perform the sector decomposition. When run it produces the directory *easy* which contains the code required to numerically evaluate the integral. The make command builds this code and produces a library. The file `integrate_easy.py` loads the integral library and evaluates the integral. The user is encouraged to copy and adapt these files to evaluate their own integrals.

Note: If the user is interested in evaluating a loop integral there are many convenience functions that make this much easier. Please see *Evaluating a Loop Integral* for more details.

In `generate_easy.py` we first import *make_package*, a function which can decompose, subtract and expand regulated integrals and write a C++ package to evaluate them. To define our integral we give it a *name* which will be used as the name of the output directory and C++ namespace. The *integration_variables* are declared along with a list of the name of the *regulators*. We must specify a list of the *requested_orders* to which *pySecDec* should expand our integral in each regulator. Here we specify `requested_orders = [0]` which instructs *make_package* to expand the integral up to and including $\mathcal{O}(\epsilon)$. Next, we declare the *polynomials_to_decompose*, here *sympy* syntax should be used.

```
from pySecDec import make_package

make_package(

name = 'easy',
integration_variables = ['x','y'],
regulators = ['eps'],

requested_orders = [0],
polynomials_to_decompose = ['(x+y)^(-2+eps)'],

)
```

Once the C++ library has been written and built we run `integrate_easy.py`. Here the library is loaded using *IntegralLibrary*. Calling the instance of *IntegralLibrary* with `easy_integral()` numerically evaluates the integral and returns the result.

```
from pySecDec.integral_interface import IntegralLibrary
from math import log

# load c++ library
easy = IntegralLibrary('easy/easy_pylink.so')

# integrate
_, _, result = easy()

# print result
print('Numerical Result:' + result)
print('Analytic Result:' + ' + (%f)*eps^-1 + (%f) + O(eps)' % (1.0,1.0-log(2.0)))
```

2.2 Evaluating a Loop Integral

A simple example of the evaluation of a loop integral with *pySecDec* is *box1L*. This example computes a one-loop box with one off-shell leg (with off-shellness `s1`) and one internal massive line (with mass squared `msq`), it is shown in Fig. 2.1.

To run the example change to the *box1L* directory and run the commands:

```
$ python generate_box1L.py
$ make -C box1L
$ python integrate_box1L.py
```

This will print the result of the integral evaluated with Mandelstam invariants `s=4.0`, `t=-0.75` and `s1=1.25`, `msq=1.0`:

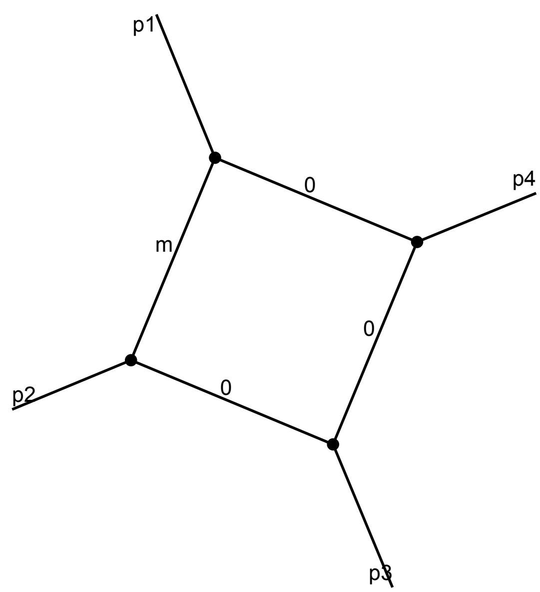

Fig. 2.1: Diagrammatic representation of *box1L*

```
eps^-2: -0.142868356275422825 - 1.63596224151119965e-6*I +/- ( 0.00118022544307414272␣
↪+ 0.000210769456586696187*I )
eps^-1: 0.639405625715768089 + 1.34277036689902802e-6*I +/- ( 0.00650722394065588166␣
↪+ 0.000971496627153705891*I )
eps^0 : -0.425514350373418893 + 1.86892487760861536*I +/- ( 0.00706834403694714484 +␣
↪0.0186497890361357298*I )
```

The file `generate_box1L.py` defines the loop integral and calls *pySecDec* to perform the sector decomposition. When run it produces the directory *box1L* which contains the code required to numerically evaluate the integral. The make command builds this code and produces a library. The file `integrate_box1L.py` loads the integral library and evaluates the integral for a specified numerical point.

The content of the python files is described in detail in the following sections. The user is encouraged to copy and adapt these files to evaluate their own loop integrals.

2.2.1 Defining a Loop Integral

To explain the input format, let us look at `generate_box1L.py` from the one-loop box example. The first two lines read

```
import pySecDec as psd
from pySecDec.loop_integral import loop_package
```

They say that the module *pySecDec* should be imported with the alias *psd*, and that the function *loop_package* from the module *loop_integral* is needed.

The following part contains the definition of the loop integral `li`:

```
li = psd.loop_integral.LoopIntegralFromGraph(
# give adjacency list and indicate whether the propagator connecting the numbered␣
↪vertices is massive or massless in the first entry of each list item.
internal_lines = [['m',[1,2]],[0,[2,3]],[0,[3,4]],[0,[4,1]]],
# contains the names of the external momenta and the label of the vertex they are␣
↪attached to
external_lines = [['p1',1],['p2',2],['p3',3],['p4',4]],

# define the kinematics and the names for the kinematic invariants
replacement_rules = [
                        ('p1*p1', 's1'),
                        ('p2*p2', 0),
                        ('p3*p3', 0),
                        ('p4*p4', 0),
                        ('p3*p2', 't/2'),
                        ('p1*p2', 's/2-s1/2'),
                        ('p1*p4', 't/2-s1/2'),
                        ('p2*p4', 's1/2-t/2-s/2'),
                        ('p3*p4', 's/2'),
                        ('m**2', 'msq')
                   ]
)
```

Here the class *LoopIntegralFromGraph* is used to Feynman parametrize the loop integral given the adjacency list. Alternatively, the class *LoopIntegralFromPropagators* can be used to construct the Feynman integral given the momentum representation.

The symbols for the kinematic invariants and the masses also need to be given as an ordered list. The ordering is important as the numerical values assigned to these list elements at the numerical evaluation stage should have the same order.

```
Mandelstam_symbols = ['s','t','s1']
mass_symbols = ['msq']
```

Next, the function `loop_package` is called. It will create a folder called *box1L*. It performs the algebraic sector decomposition steps and writes a package containing the C++ code for the numerical evaluation. The argument *requested_order* specifies the order in the regulator to which the integral should be expanded. For a complete list of possible options see `loop_package`.

```
loop_package(

name = 'box1L',

loop_integral = li,

real_parameters = Mandelstam_symbols + mass_symbols,

# the highest order of the final epsilon expansion --> change this value to whatever␣
↪you think is appropriate
requested_order = 0,

# the optimization level to use in FORM (can be 0, 1, 2, 3, 4)
form_optimization_level = 2,

# the WorkSpace parameter for FORM
form_work_space = '100M',

# the method to be used for the sector decomposition
# valid values are ``iterative`` or ``geometric`` or ``geometric_ku``
decomposition_method = 'iterative',
# if you choose ``geometric[_ku]`` and 'normaliz' is not in your
# $PATH, you can set the path to the 'normaliz' command-line
# executable here
#normaliz_executable='/path/to/normaliz',

)
```

2.2.2 Building the C++ Library

After running the python script *generate_box1L.py* the folder *box1L* is created and should contain the following files and subdirectories

```
Makefile    Makefile.conf    README    box1L.hpp    codegen    integrate_box1L.cpp    ␣
↪cuda_integrate_box1L.cpp    pylink    src
```

in the folder *box1L*, typing

```
$ make
```

will create the static library `libbox1L.a` and `box1L_pylink.so` which can be linked to external programs. The `make` command can also be run in parallel by using the `-j` option. The number of threads each instance of `tform` uses can be set via the environment variable *FORMTHREADS*.

New in version 1.4: The environment variable *FORMOPT* sets FORM's code optimization level. If not set, the value that was passed to `make_package` or `loop_package` is used.

To build the dynamic library `libbox1L.so` set `dynamic` as build target:

```
$ make dynamic
```

The code generation with FORM without subsequent compilation can be run by setting `source` as build target.

To build the library with *nvcc* for GPU support, type

```
$ CXX=nvcc SECDEC_WITH_CUDA=sm_XX make
```

where `sm_XX` must be replaced by the target GPU architechtures, see the arch option of NVCC.

To evaluate the integral numerically a program can call one of these libraries. How to do this interactively or via a python script is explained in the section *Python Interface*. Alternatively, a C++ program can be produced as explained in the section *C++ Interface*.

2.2.3 Python Interface (basic)

To evaluate the integral for a given numerical point we can use `integrate_box1L.py`. First it imports the necessary python packages and loads the C++ library.

```
from __future__ import print_function
from pySecDec.integral_interface import IntegralLibrary
import sympy as sp

# load c++ library
box1L = IntegralLibrary('box1L/box1L_pylink.so')
```

Next, an integrator is configured for the numerical integration. The full list of available integrators and their options is given in *`integral_interface`*.

```
# choose integrator
box.use_Vegas(flags=2) # ``flags=2``: verbose --> see Cuba manual
```

If you want to use GPUs, change to the *`CudaQmc`* integrator. For example, to run on all available GPUs and CPU cores using the Korobov transform with weight 3, change the above lines to

```
# choose integrator
box.use_Qmc(transform='Korobov3')
```

Calling the `box` library numerically evaluates the integral. Note that the order of the real parameters must match that specified in `generate_box1L.py`. A list of possible settings for the library, in particular details of how to set the contour deformation parameters, is given in *`IntegralLibrary`*. To change the accuracy settings of the integration, the most important parameters are `epsrel`, `epsabs` and `maxeval`, which can be added to the integrator argument list:

```
# choose integrator
box.use_Vegas(flags=2,epsrel=0.01, epsabs=1e-07, maxeval=1000000)
```

```
# integrate
str_integral_without_prefactor, str_prefactor, str_integral_with_prefactor =␣
↪box1L(real_parameters=[4.0, -0.75, 1.25, 1.0])
```

In case of a sign check error (sign_check_error), the arguments `number_of_presamples`, `deformation_parameters_maximum`, and `deformation_parameters_minimum` as described in *`IntegralLibrary`* can be used to modify the contour. At this point the string `str_integral_with_prefactor` contains the full result of the integral and can be manipulated as required. In the `integrate_box1L.py` an example is shown how to parse the expression with *sympy* and access individual orders of the regulator.

Note: Instead of parsing the result, it can simply be printed with the line `print(str_integral_with_prefactor)`.

```
# convert complex numbers from c++ to sympy notation
str_integral_with_prefactor = str_integral_with_prefactor.replace(',','+I*')
str_prefactor = str_prefactor.replace(',','+I*')
str_integral_without_prefactor = str_integral_without_prefactor.replace(',','+I*')

# convert result to sympy expressions
integral_with_prefactor = sp.sympify(str_integral_with_prefactor.replace('+/-',
↪'*value+error*'))
integral_with_prefactor_err = sp.sympify(str_integral_with_prefactor.replace('+/-',
↪'*value+error*'))
prefactor = sp.sympify(str_prefactor)
integral_without_prefactor = sp.sympify(str_integral_without_prefactor.replace('+/-',
↪'*value+error*'))
integral_without_prefactor_err = sp.sympify(str_integral_without_prefactor.replace('+/
↪-','*value+error*'))

# examples how to access individual orders
print('Numerical Result')
print('eps^-2:', integral_with_prefactor.coeff('eps',-2).coeff('value'), '+/- (',␣
↪integral_with_prefactor_err.coeff('eps',-2).coeff('error'), ')')
print('eps^-1:', integral_with_prefactor.coeff('eps',-1).coeff('value'), '+/- (',␣
↪integral_with_prefactor_err.coeff('eps',-1).coeff('error'), ')')
print('eps^0 :', integral_with_prefactor.coeff('eps',0).coeff('value'), '+/- (',␣
↪integral_with_prefactor_err.coeff('eps',0).coeff('error'), ')')
```

An example of how to loop over several kinematic points is shown in the example *multiple_kinematic_points.py*.

2.2.4 C++ Interface (advanced)

Usually it is easier to obtain a numerical result using the *Python Interface*. However, the library can also be used directly from C++. Inside the generated *box1L* folder the file `integrate_box1L.cpp` demonstrates this.

The function `print_integral_info` shows how to access the important variables of the integral library.

In the `main` function a kinematic point must be specified by setting the `real_parameters` variable, for example:

```
int main()
{
//  User Specified Phase-space point
    const std::vector<box1L::real_t> real_parameters = {4.0, -0.75, 1.25, 1.0}; //␣
↪EDIT: kinematic point specified here
    const std::vector<box1L::complex_t> complex_parameters = {  };
```

The `name::make_integrands()` function returns an `secdecutil::IntegrandContainer` for each sector and regulator order:

```
//  Generate the integrands (optimization of the contour if applicable)
    const std::vector<box1L::nested_series_t<box1L::integrand_t>> sector_integrands =␣
↪box1L::make_integrands(real_parameters, complex_parameters);
```

The contour deformation has to be adjusted in case of a sign check error (sign_check_error). This can be done via additional arguments to *name::make_integrands()*. The sectors can be added before integration:

```
//  Add integrands of sectors (together flag)
    const box1L::nested_series_t<box1L::integrand_t> all_sectors =␣
↪std::accumulate(++sector_integrands.begin(), sector_integrands.end(), *sector_
↪integrands.begin() );
```

An *secdecutil::Integrator* is constructed and its parameters are set:

```
//  Integrate
    secdecutil::cuba::Vegas<box1L::integrand_return_t> integrator;
    integrator.flags = 2; // verbose output --> see cuba manual
```

To numerically integrate the functions the *secdecutil::Integrator::integrate()* function is applied to each *secdecutil::IntegrandContainer* using *secdecutil::deep_apply()*:

```
const box1L::nested_series_t<secdecutil::UncorrelatedDeviation<box1L::integrand_
↪return_t>> result_all = secdecutil::deep_apply( all_sectors, integrator.integrate );
```

The remaining lines print the result:

```
    std::cout << "------------" << std::endl << std::endl;

    std::cout << "-- integral info -- " << std::endl;
    print_integral_info();
    std::cout << std::endl;

    std::cout << "-- integral without prefactor -- " << std::endl;
    std::cout << result_all << std::endl << std::endl;

    std::cout << "-- prefactor -- " << std::endl;
    const box1L::nested_series_t<box1L::integrand_return_t> prefactor =␣
↪box1L::prefactor(real_parameters, complex_parameters);
    std::cout << prefactor << std::endl << std::endl;

    std::cout << "-- full result (prefactor*integral) -- " << std::endl;
    std::cout << prefactor*result_all << std::endl;
    return 0;
}
```

After editing the `real_parameters` as described above the C++ program can be built and executed with the commands

```
$ make integrate_box1L
$ ./integrate_box1L
```

New in version 1.4.

The similar template file `cuda_integrate_box1L.cpp` provides an example to run on GPUs. The main differences are in the lines that generate, add, and integrate the integrands. Rather than *name::make_integrands()*, *name::make_cuda_integrands()* is called:

```
// Generate the integrands (optimization of the contour if applicable)
const std::vector<box1L::nested_series_t<box1L::cuda_integrand_t>> sector_integrands␣
↪= box1L::make_cuda_integrands(real_parameters, complex_parameters);
```

If the integrands are added together before integration, the sum command is as follows:

```
// Add integrands of sectors (together flag)
const box1L::nested_series_t<box1L::cuda_together_integrand_t> all_sectors =
    std::accumulate(++sector_integrands.begin(), sector_integrands.end(), box1L::cuda_
↪together_integrand_t()+*sector_integrands.begin());
```

Note the conversion from *name::cuda_integrand_t* to *name::cuda_together_integrand_t*. The CUDA-capable version of the Qmc integrator takes additional the template arguments `box1L::maximal_number_of_integration_variables`, `integrators::transforms::Korobov<3>::type`, and *name::cuda_integrand_t*:

```
// Integrate
secdecutil::integrators::Qmc<
                                box1L::integrand_return_t,
                                box1L::maximal_number_of_integration_variables,
                                integrators::transforms::Korobov<3>::type, // EDIT:␣
↪integral transform specified to "Korobov<3>"
                                box1L::cuda_together_integrand_t
                            > integrator;
integrator.verbosity = 1;
const box1L::nested_series_t<secdecutil::UncorrelatedDeviation<box1L::integrand_
↪return_t>> result_all = secdecutil::deep_apply( all_sectors, integrator.integrate );
```

If the integrands are integrated separately, *name::cuda_together_integrand_t* should be changed to *name::cuda_integrand_t*. If your integral is higher than seven dimensional, changing the integral transform to `integrators::transforms::Baker::type` may improve the accuracy of the result. For further options of the integrator we refer to Section 4.5.2.

2.3 List of Examples

Here we list the available examples. For more details regarding each example see [PSD17] and [PSD18].

easy:	a simple parametric integral, described in Section 2.1
easy_cuda:	the same integral as in **easy** but computed on GPUs with CUDA
box1L:	a simple 1-loop, 4-point, 4-propagator integral, described in Section 2.2
triangle2L:	a 2-loop, 3-point, 6-propagator diagram, also known as *P126*
box2L_numerator:	a massless planar on-shell 2-loop, 4-point, 7-propagator box with a numerator, either defined as an inverse propagator `box2L_invprop.py` or in terms of contracted Lorentz vectors `box2L_contracted_tensor.py`
pentabox_fin:	a 2-loop, 5-point, 8-propagator diagram, evaluated in $6-2\epsilon$ dimensions where it is finite
triangle3L:	a 2-loop, 3-point, 7-propagator integral, demonstrates that the symmetry finder can significantly reduce the number of sectors
formfactor4L:	a single-scale 4-loop 3-point integral in $6-2\epsilon$ dimensions
bubble6L:	a single-scale 6-loop 2-point integral, evaluated at a Euclidean phase-space point
elliptic2L_euclidean:	an integral known to contain elliptic functions, evaluated at a Euclidean phase-space point
elliptic2L_physical:	an integral known to contain elliptic functions, evaluated at a physical phase-space point
banana_3mass:	a 3-loop 2-point integral with three different internal masses known to contain hyperelliptic functions, evaluated at a physical phase-space point
hyperelliptic:	a 2-loop 4-point nonplanar integral known to contain hyperelliptic functions, evaluated at a physical phase-space point
triangle2L_split:	a 2-loop, 3-point, 6-propagator integral without a Euclidean region due to special kinematics
Nbox2L_split:	three 2-loop, 4-point, 5-propagator integrals that need `split=True` due to special kinematics
hypergeo5F4:	a general dimensionally regulated parameter integral
4photon1L_amplitude:	calculation of the 4-photon amplitude, showing how to use *pySecDec* as an integral library in a larger context
two_regulators:	an integral involving poles in two different regulators.
userdefined_cpp:	a collection of examples demonstrating how to combine polynomials to be decomposed with other user-defined functions

CHAPTER 3

Overview

pySecDec consists of several modules that provide functions and classes for specific purposes. In this overview, we present only the most important aspects of selected modules. These are exactly the modules necessary to set up the algebraic computation of a Feynman loop integral requisite for the numerical evaluation. For detailed instruction of a specific function or class, please be referred to the *reference guide*.

3.1 The Algebra Module

The *algebra* module implements a very basic computer algebra system. *pySecDec* uses both *sympy* and *numpy*. Although *sympy* in principle provides everything we need, it is way too slow for typical applications. That is because *sympy* is completely written in *python* without making use of any precompiled functions. *pySecDec*'s *algebra* module uses the in general faster *numpy* function wherever possible.

3.1.1 Polynomials

Since sector decomposition is an algorithm that acts on polynomials, we start with the key class `Polynomial`. As the name suggests, the `Polynomial` class is a container for multivariate polynomials, i.e. functions of the form:

$$\sum_i C_i \prod_j x_j^{\alpha_{ij}}$$

A multivariate polynomial is completely determined by its *coefficients* C_i and the exponents α_{ij}. The `Polynomial` class stores these in two arrays:

```
>>> from pySecDec.algebra import Polynomial
>>> poly = Polynomial([[1,0], [0,2]], ['A', 'B'])
>>> poly
 + (A)*x0 + (B)*x1**2
>>> poly.expolist
array([[1, 0],
       [0, 2]])
```

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```
>>> poly.coeffs
array([A, B], dtype=object)
```

It is also possible to instantiate the *Polynomial* by its algebraic representation:

```
>>> poly2 = Polynomial.from_expression('A*x0 + B*x1**2', ['x0','x1'])
>>> poly2
 + (A)*x0 + (B)*x1**2
>>> poly2.expolist
array([[1, 0],
       [0, 2]])
>>> poly2.coeffs
array([A, B], dtype=object)
```

Note that the second argument of *Polynomial.from_expression()* defines the variables x_j.

Within the *Polynomial* class, basic operations are implemented:

```
>>> poly + 1
 + (1) + (B)*x1**2 + (A)*x0
>>> 2 * poly
 + (2*A)*x0 + (2*B)*x1**2
>>> poly + poly
 + (2*B)*x1**2 + (2*A)*x0
>>> poly * poly
 + (B**2)*x1**4 + (2*A*B)*x0*x1**2 + (A**2)*x0**2
>>> poly ** 2
 + (B**2)*x1**4 + (2*A*B)*x0*x1**2 + (A**2)*x0**2
```

3.1.2 General Expressions

In order to perform the *pySecDec.subtraction* and *pySecDec.expansion*, we have to introduce more complex algebraic constructs.

General expressions can be entered in a straightforward way:

```
>>> from pySecDec.algebra import Expression
>>> log_of_x = Expression('log(x)', ['x'])
>>> log_of_x
log( + (1)*x)
```

All expressions in the context of this *algebra* module are based on extending or combining the *Polynomials* introduced *above*. In the example above, `log_of_x` is a *LogOfPolynomial*, which is a derived class from *Polynomial*:

```
>>> type(log_of_x)
<class 'pySecDec.algebra.LogOfPolynomial'>
>>> isinstance(log_of_x, Polynomial)
True
>>> log_of_x.expolist
array([[1]])
>>> log_of_x.coeffs
array([1], dtype=object)
```

We have seen an *extension* to the *Polynomial* class, now let us consider a *combination*:

```
>>> more_complex_expression = log_of_x * log_of_x
>>> more_complex_expression
(log( + (1)*x)) * (log( + (1)*x))
```

We just introduced the *Product* of two *LogOfPolynomials*:

```
>>> type(more_complex_expression)
<class 'pySecDec.algebra.Product'>
```

As suggested before, the *Product* combines two *Polynomials*. They are accessible through the `factors`:

```
>>> more_complex_expression.factors[0]
log( + (1)*x)
>>> more_complex_expression.factors[1]
log( + (1)*x)
>>> type(more_complex_expression.factors[0])
<class 'pySecDec.algebra.LogOfPolynomial'>
>>> type(more_complex_expression.factors[1])
<class 'pySecDec.algebra.LogOfPolynomial'>
```

Important: When working with this *algebra* module, it is important to understand that **everything** is based on the class *Polynomial*.

To emphasize the importance of the above statement, consider the following code:

```
>>> expression1 = Expression('x*y', ['x', 'y'])
>>> expression2 = Expression('x*y', ['x'])
>>> type(expression1)
<class 'pySecDec.algebra.Polynomial'>
>>> type(expression2)
<class 'pySecDec.algebra.Polynomial'>
>>> expression1
 + (1)*x*y
>>> expression2
 + (y)*x
```

Although `expression1` and `expression2` are mathematically identical, they are treated differently by the *algebra* module. In `expression1`, both, `x` and `y`, are considered as variables of the *Polynomial*. In contrast, `y` is treated as *coefficient* in `expression2`:

```
>>> expression1.expolist
array([[1, 1]])
>>> expression1.coeffs
array([1], dtype=object)
>>> expression2.expolist
array([[1]])
>>> expression2.coeffs
array([y], dtype=object)
```

The second argument of the function *Expression* controls how the variables are distributed among the coefficients and the variables in the underlying *Polynomials*. Keep that in mind in order to avoid confusion. One can always check which symbols are considered as variables by asking for the `symbols`:

```
>>> expression1.symbols
[x, y]
```

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```
>>> expression2.symbols
[x]
```

3.2 Feynman Parametrization of Loop Integrals

The primary purpose of *pySecDec* is the numerical calculation of loop integrals as they arise in fixed order calculations in quantum field theories. In the first step of our approach, the loop integral is converted from the momentum representation to the Feynman parameter representation, see for example [Hei08] (Chapter 3).

The module `pySecDec.loop_integral` implements exactly that conversion. The most basic use is to calculate the first and the second Symanzik polynomial `U` and `F`, respectively, from the propagators of a loop integral.

3.2.1 One Loop Bubble

To calculate `U` and `F` of the one loop bubble, type the following commands:

```
>>> from pySecDec.loop_integral import LoopIntegralFromPropagators
>>> propagators = ['k**2', '(k - p)**2']
>>> loop_momenta = ['k']
>>> one_loop_bubble = LoopIntegralFromPropagators(propagators, loop_momenta)
>>> one_loop_bubble.U
 + (1)*x0 + (1)*x1
>>> one_loop_bubble.F
 + (-p**2)*x0*x1
```

The example above among other useful features is also stated in the full documenation of `LoopIntegralFromPropagators()` in the reference guide.

3.2.2 Two Loop Planar Box with Numerator

Consider the propagators of the two loop planar box:

```
>>> propagators = ['k1**2','(k1+p2)**2',
...                '(k1-p1)**2','(k1-k2)**2',
...                '(k2+p2)**2','(k2-p1)**2',
...                '(k2+p2+p3)**2']
>>> loop_momenta = ['k1','k2']
```

We could now instantiate the `LoopIntegral` just like *before*. However, let us consider an additional numerator:

```
>>> numerator = 'k1(mu)*k1(mu) + 2*k1(mu)*p3(mu) + p3(mu)*p3(mu)' # (k1 + p3) ** 2
```

In order to unambiguously define the loop integral, we must state which symbols denote the `Lorentz_indices` (just `mu` in this case here) and the `external_momenta`:

```
>>> external_momenta = ['p1','p2','p3','p4']
>>> Lorentz_indices=['mu']
```

With that, we can Feynman parametrize the two loop box with a numerator:

```
>>> box = LoopIntegralFromPropagators(propagators, loop_momenta, external_momenta,
...                                   numerator=numerator, Lorentz_indices=Lorentz_
↪indices)
>>> box.U
 + (1)*x3*x6 + (1)*x3*x5 + (1)*x3*x4 + (1)*x2*x6 + (1)*x2*x5 + (1)*x2*x4 + (1)*x2*x3
↪+ (1)*x1*x6 + (1)*x1*x5 + (1)*x1*x4 + (1)*x1*x3 + (1)*x0*x6 + (1)*x0*x5 + (1)*x0*x4
↪+ (1)*x0*x3
>>> box.F
 + (-p1**2 - 2*p1*p2 - 2*p1*p3 - p2**2 - 2*p2*p3 - p3**2)*x3*x5*x6 + (-
↪p3**2)*x3*x4*x6 + (-p1**2 - 2*p1*p2 - p2**2)*x3*x4*x5 + (-p1**2 - 2*p1*p2 - 2*p1*p3
↪- p2**2 - 2*p2*p3 - p3**2)*x2*x5*x6 + (-p3**2)*x2*x4*x6 + (-p1**2 - 2*p1*p2 -
↪p2**2)*x2*x4*x5 + (-p1**2 - 2*p1*p2 - 2*p1*p3 - p2**2 - 2*p2*p3 - p3**2)*x2*x3*x6 +
↪(-p1**2 - 2*p1*p2 - p2**2)*x2*x3*x4 + (-p1**2 - 2*p1*p2 - 2*p1*p3 - p2**2 - 2*p2*p3
↪- p3**2)*x1*x5*x6 + (-p3**2)*x1*x4*x6 + (-p1**2 - 2*p1*p2 - p2**2)*x1*x4*x5 + (-
↪p3**2)*x1*x3*x6 + (-p1**2 - 2*p1*p2 - p2**2)*x1*x3*x5 + (-p1**2 - 2*p1*p2 -
↪p2**2)*x1*x2*x6 + (-p1**2 - 2*p1*p2 - p2**2)*x1*x2*x5 + (-p1**2 - 2*p1*p2 -
↪p2**2)*x1*x2*x4 + (-p1**2 - 2*p1*p2 - p2**2)*x1*x2*x3 + (-p1**2 - 2*p1*p2 - 2*p1*p3
↪- p2**2 - 2*p2*p3 - p3**2)*x0*x5*x6 + (-p3**2)*x0*x4*x6 + (-p1**2 - 2*p1*p2 -
↪p2**2)*x0*x4*x5 + (-p2**2 - 2*p2*p3 - p3**2)*x0*x3*x6 + (-p1**2)*x0*x3*x5 + (-
↪p2**2)*x0*x3*x4 + (-p1**2)*x0*x2*x6 + (-p1**2)*x0*x2*x5 + (-p1**2)*x0*x2*x4 + (-
↪p1**2)*x0*x2*x3 + (-p2**2)*x0*x1*x6 + (-p2**2)*x0*x1*x5 + (-p2**2)*x0*x1*x4 + (-
↪p2**2)*x0*x1*x3
>>> box.numerator
 + (2*eps*p3(mu)**2 + 2*p3(mu)**2)*U**2 + (eps - 2)*x6*F + (eps - 2)*x5*F + (eps -
↪2)*x4*F + (eps - 2)*x3*F + (-4*eps*p2(mu)*p3(mu) - 4*eps*p3(mu)**2 -
↪4*p2(mu)*p3(mu) - 4*p3(mu)**2)*x3*x6*U + (4*eps*p1(mu)*p3(mu) +
↪4*p1(mu)*p3(mu))*x3*x5*U + (-4*eps*p2(mu)*p3(mu) - 4*p2(mu)*p3(mu))*x3*x4*U +
↪(2*eps*p2(mu)**2 + 4*eps*p2(mu)*p3(mu) + 2*eps*p3(mu)**2 + 2*p2(mu)**2 +
↪4*p2(mu)*p3(mu) + 2*p3(mu)**2)*x3**2*x6**2 + (-4*eps*p1(mu)*p2(mu) -
↪4*eps*p1(mu)*p3(mu) - 4*p1(mu)*p2(mu) - 4*p1(mu)*p3(mu))*x3**2*x5*x6 +
↪(2*eps*p1(mu)**2 + 2*p1(mu)**2)*x3**2*x5**2 + (4*eps*p2(mu)**2 +
↪4*eps*p2(mu)*p3(mu) + 4*p2(mu)**2 + 4*p2(mu)*p3(mu))*x3**2*x4*x6 + (-
↪4*eps*p1(mu)*p2(mu) - 4*p1(mu)*p2(mu))*x3**2*x4*x5 + (2*eps*p2(mu)**2 +
↪2*p2(mu)**2)*x3**2*x4**2 + (4*eps*p1(mu)*p3(mu) + 4*p1(mu)*p3(mu))*x2*x6*U +
↪(4*eps*p1(mu)*p3(mu) + 4*p1(mu)*p3(mu))*x2*x5*U + (4*eps*p1(mu)*p3(mu) +
↪4*p1(mu)*p3(mu))*x2*x4*U + (4*eps*p1(mu)*p3(mu) + 4*p1(mu)*p3(mu))*x2*x3*U + (-
↪4*eps*p1(mu)*p2(mu) - 4*eps*p1(mu)*p3(mu) - 4*p1(mu)*p2(mu) -
↪4*p1(mu)*p3(mu))*x2*x3*x6**2 + (4*eps*p1(mu)**2 - 4*eps*p1(mu)*p2(mu) -
↪4*eps*p1(mu)*p3(mu) + 4*p1(mu)**2 - 4*p1(mu)*p2(mu) - 4*p1(mu)*p3(mu))*x2*x3*x5*x6
↪+ (4*eps*p1(mu)**2 + 4*p1(mu)**2)*x2*x3*x5**2 + (-8*eps*p1(mu)*p2(mu) -
↪4*eps*p1(mu)*p3(mu) - 8*p1(mu)*p2(mu) - 4*p1(mu)*p3(mu))*x2*x3*x4*x6 +
↪(4*eps*p1(mu)**2 - 4*eps*p1(mu)*p2(mu) + 4*p1(mu)**2 - 4*p1(mu)*p2(mu))*x2*x3*x4*x5
↪+ (-4*eps*p1(mu)*p2(mu) - 4*p1(mu)*p2(mu))*x2*x3*x4**2 + (-4*eps*p1(mu)*p2(mu) -
↪4*eps*p1(mu)*p3(mu) - 4*p1(mu)*p2(mu) - 4*p1(mu)*p3(mu))*x2*x3**2*x6 +
↪(4*eps*p1(mu)**2 + 4*p1(mu)**2)*x2*x3**2*x5 + (-4*eps*p1(mu)*p2(mu) -
↪4*p1(mu)*p2(mu))*x2*x3**2*x4 + (2*eps*p1(mu)**2 + 2*p1(mu)**2)*x2**2*x6**2 +
↪(4*eps*p1(mu)**2 + 4*p1(mu)**2)*x2**2*x5*x6 + (2*eps*p1(mu)**2 +
↪2*p1(mu)**2)*x2**2*x5**2 + (4*eps*p1(mu)**2 + 4*p1(mu)**2)*x2**2*x4*x6 +
↪(4*eps*p1(mu)**2 + 4*p1(mu)**2)*x2**2*x4*x5 + (2*eps*p1(mu)**2 +
↪2*p1(mu)**2)*x2**2*x4**2 + (4*eps*p1(mu)**2 + 4*p1(mu)**2)*x2**2*x3*x6 +
↪(4*eps*p1(mu)**2 + 4*p1(mu)**2)*x2**2*x3*x5 + (4*eps*p1(mu)**2 +
↪4*p1(mu)**2)*x2**2*x3*x4 + (2*eps*p1(mu)**2 + 2*p1(mu)**2)*x2**2*x3**2 + (-
↪4*eps*p2(mu)*p3(mu) - 4*p2(mu)*p3(mu))*x1*x6*U + (-4*eps*p2(mu)*p3(mu) -
↪4*p2(mu)*p3(mu))*x1*x5*U + (-4*eps*p2(mu)*p3(mu) - 4*p2(mu)*p3(mu))*x1*x4*U + (-
↪4*eps*p2(mu)*p3(mu) - 4*p2(mu)*p3(mu))*x1*x3*U + (4*eps*p2(mu)**2 +
↪4*eps*p2(mu)*p3(mu) + 4*p2(mu)**2 + 4*p2(mu)*p3(mu))*x1*x3*x6**2 + (-
↪4*eps*p1(mu)*p2(mu) + 4*eps*p2(mu)**2 + 4*eps*p2(mu)*p3(mu) - 4*p1(mu)*p2(mu) +
↪4*p2(mu)**2 + 4*p2(mu)*p3(mu))*x1*x3*x5*x6 + (-4*eps*p1(mu)*p2(mu) -
↪4*p1(mu)*p2(mu))*x1*x3*x5**2 + (8*eps*p2(mu)**2 + 4*eps*p2(mu)*p3(mu) + 8*p2(mu)**2
↪+ 4*p2(mu)*p3(mu))*x1*x3*x4*x6 + (-4*eps*p1(mu)*p2(mu) + 4*eps*p2(mu)**2 -
↪4*p1(mu)*p2(mu) + 4*p2(mu)**2)*x1*x3*x4*x5 + (4*eps*p2(mu)**2 +
↪4*p2(mu)**2)*x1*x3*x4**2 + (4*eps*p2(mu)**2 + 4*eps*p2(mu)*p3(mu) + 4*p2(mu)**2 +
↪4*p2(mu)*p3(mu))*x1*x3**2*x6 + (-4*eps*p1(mu)*p2(mu) - 4*p1(mu)*p2(mu))*x1*x3**2*x5
↪+ (4*eps*p2(mu)**2 + 4*p2(mu)**2)*x1*x3**2*x4 + (-4*eps*p1(mu)*p2(mu) -
↪4*p1(mu)*p2(mu))*x1*x2*x6**2 + (-8*eps*p1(mu)*p2(mu) - 8*p1(mu)*p2(mu))*x1*x2*x5*x6
```

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We can also generate the output in terms of Mandelstam invariants:

```
>>> replacement_rules = [
...                         ('p1*p1', 0),
...                         ('p2*p2', 0),
...                         ('p3*p3', 0),
...                         ('p4*p4', 0),
...                         ('p1*p2', 's/2'),
...                         ('p2*p3', 't/2'),
...                         ('p1*p3', '-s/2-t/2')
...                     ]
>>> box = LoopIntegralFromPropagators(propagators, loop_momenta, external_momenta,
...                                   numerator=numerator, Lorentz_indices=Lorentz_
↪indices,
...                                   replacement_rules=replacement_rules)
>>> box.U
 + (1)*x3*x6 + (1)*x3*x5 + (1)*x3*x4 + (1)*x2*x6 + (1)*x2*x5 + (1)*x2*x4 + (1)*x2*x3␣
↪+ (1)*x1*x6 + (1)*x1*x5 + (1)*x1*x4 + (1)*x1*x3 + (1)*x0*x6 + (1)*x0*x5 + (1)*x0*x4␣
↪+ (1)*x0*x3
>>> box.F
 + (-s)*x3*x4*x5 + (-s)*x2*x4*x5 + (-s)*x2*x3*x4 + (-s)*x1*x4*x5 + (-s)*x1*x3*x5 + (-
↪s)*x1*x2*x6 + (-s)*x1*x2*x5 + (-s)*x1*x2*x4 + (-s)*x1*x2*x3 + (-s)*x0*x4*x5 + (-
↪t)*x0*x3*x6
>>> box.numerator
 + (eps - 2)*x6*F + (eps - 2)*x5*F + (eps - 2)*x4*F + (eps - 2)*x3*F + (-2*eps*t -␣
↪2*t)*x3*x6*U + (4*eps*(-s/2 - t/2) - 2*s - 2*t)*x3*x5*U + (-2*eps*t - 2*t)*x3*x4*U␣
↪+ (2*eps*t + 2*t)*x3**2*x6**2 + (-2*eps*s - 4*eps*(-s/2 - t/2) + 2*t)*x3**2*x5*x6 +␣
↪(2*eps*t + 2*t)*x3**2*x4*x6 + (-2*eps*s - 2*s)*x3**2*x4*x5 + (4*eps*(-s/2 - t/2) -␣
↪2*s - 2*t)*x2*x6*U + (4*eps*(-s/2 - t/2) - 2*s - 2*t)*x2*x5*U + (4*eps*(-s/2 - t/2)␣
↪- 2*s - 2*t)*x2*x4*U + (4*eps*(-s/2 - t/2) - 2*s - 2*t)*x2*x3*U + (-2*eps*s -␣
↪4*eps*(-s/2 - t/2) + 2*t)*x2*x3*x6**2 + (-2*eps*s - 4*eps*(-s/2 - t/2) +␣
↪2*t)*x2*x3*x5*x6 + (-4*eps*s - 4*eps*(-s/2 - t/2) - 2*s + 2*t)*x2*x3*x4*x6 + (-
↪2*eps*s - 2*s)*x2*x3*x4*x5 + (-2*eps*s - 2*s)*x2*x3*x4**2 + (-2*eps*s - 4*eps*(-s/2␣
↪- t/2) + 2*t)*x2*x3**2*x6 + (-2*eps*s - 2*s)*x2*x3**2*x4 + (-2*eps*t - 2*t)*x1*x6*U␣
↪+ (-2*eps*t - 2*t)*x1*x5*U + (-2*eps*t - 2*t)*x1*x4*U + (-2*eps*t - 2*t)*x1*x3*U +␣
↪(2*eps*t + 2*t)*x1*x3*x6**2 + (-2*eps*s + 2*eps*t - 2*s + 2*t)*x1*x3*x5*x6 + (-
↪2*eps*s - 2*s)*x1*x3*x5**2 + (2*eps*t + 2*t)*x1*x3*x4*x6 + (-2*eps*s -␣
↪2*s)*x1*x3*x4*x5 + (2*eps*t + 2*t)*x1*x3**2*x6 + (-2*eps*s - 2*s)*x1*x3**2*x5 + (-
↪2*eps*s - 2*s)*x1*x2*x6**2 + (-4*eps*s - 4*s)*x1*x2*x5*x6 + (-2*eps*s -␣
↪2*s)*x1*x2*x5**2 + (-4*eps*s - 4*s)*x1*x2*x4*x6 + (-4*eps*s - 4*s)*x1*x2*x4*x5 + (-
↪2*eps*s - 2*s)*x1*x2*x4**2 + (-4*eps*s - 4*s)*x1*x2*x3*x6 + (-4*eps*s -␣
↪4*s)*x1*x2*x3*x5 + (-4*eps*s - 4*s)*x1*x2*x3*x4 + (-2*eps*s - 2*s)*x1*x2*x3**2
```

3.3 Sector Decomposition

The sector decomposition algorithm aims to factorize the polynomials P_i as products of a monomial and a polynomial with nonzero constant term:

$$P_i(\{x_j\}) \longmapsto \prod_j x_j^{\alpha_j} \left(const + p_i(\{x_j\})\right).$$

Factorizing polynomials in that way by expoliting integral transformations is the first step in an algorithm for solving dimensionally regulated integrals of the form

$$\int_0^1 \prod_{i,j} P_i(\{x_j\})^{\beta_i}\, dx_j.$$

The iterative sector decomposition splits the integral and remaps the integration domain until all polynomials P_i in all arising integrals (called *sectors*) have the desired form $const + polynomial$. An introduction to the sector decomposition approach can be found in [Hei08].

To demonstrate the *pySecDec.decomposition* module, we decompose the polynomials

```
>>> p1 = Polynomial.from_expression('x + A*y', ['x','y','z'])
>>> p2 = Polynomial.from_expression('x + B*y*z', ['x','y','z'])
```

Let us first focus on the iterative decomposition of p1. In the *pySecDec* framework, we first have to pack p1 into a *Sector*:

```
>>> from pySecDec.decomposition import Sector
>>> initial_sector = Sector([p1])
>>> print(initial_sector)
Sector:
Jacobian= + (1)
cast=[( + (1)) * ( + (1)*x + (A)*y)]
other=[]
```

We can now run the iterative decomposition and take a look at the decomposed sectors:

```
>>> from pySecDec.decomposition.iterative import iterative_decomposition
>>> decomposed_sectors = iterative_decomposition(initial_sector)
>>> for sector in decomposed_sectors:
...     print(sector)
...     print('\n')
...
Sector:
Jacobian= + (1)*x
cast=[( + (1)*x) * ( + (1) + (A)*y)]
other=[]


Sector:
Jacobian= + (1)*y
cast=[( + (1)*y) * ( + (1)*x + (A))]
other=[]
```

The decomposition of p2 needs two iterations and yields three sectors:

```
>>> initial_sector = Sector([p2])
>>> decomposed_sectors = iterative_decomposition(initial_sector)
>>> for sector in decomposed_sectors:
...     print(sector)
...     print('\n')
...
Sector:
Jacobian= + (1)*x
cast=[( + (1)*x) * ( + (1) + (B)*y*z)]
other=[]
```

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```
Sector:
Jacobian= + (1)*x*y
cast=[( + (1)*x*y) * ( + (1) + (B)*z)]
other=[]


Sector:
Jacobian= + (1)*y*z
cast=[( + (1)*y*z) * ( + (1)*x + (B))]
other=[]
```

Note that we declared `z` as a variable for sector `p1` evne though it does not depend on it. This declaration is necessary if we want to simultaneously decompose `p1` and `p2`:

```
>>> initial_sector = Sector([p1, p2])
>>> decomposed_sectors = iterative_decomposition(initial_sector)
>>> for sector in decomposed_sectors:
...     print(sector)
...     print('\n')
...
Sector:
Jacobian= + (1)*x
cast=[( + (1)*x) * ( + (1) + (A)*y), ( + (1)*x) * ( + (1) + (B)*y*z)]
other=[]


Sector:
Jacobian= + (1)*x*y
cast=[( + (1)*y) * ( + (1)*x + (A)), ( + (1)*x*y) * ( + (1) + (B)*z)]
other=[]


Sector:
Jacobian= + (1)*y*z
cast=[( + (1)*y) * ( + (1)*x*z + (A)), ( + (1)*y*z) * ( + (1)*x + (B))]
other=[]
```

We just fully decomposed `p1` and `p2`. In some cases, one may want to bring one polynomial, say `p1`, into standard form, but not neccessarily the other. For that purpose, the *Sector* can take a second argument. In the following code example, we bring `p1` into standard form, apply all transformations to `p2` as well, but stop before `p2` is fully decomposed:

```
>>> initial_sector = Sector([p1], [p2])
>>> decomposed_sectors = iterative_decomposition(initial_sector)
>>> for sector in decomposed_sectors:
...     print(sector)
...     print('\n')
...
Sector:
Jacobian= + (1)*x
cast=[( + (1)*x) * ( + (1) + (A)*y)]
other=[ + (1)*x + (B)*x*y*z]


Sector:
```

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```
Jacobian= + (1)*y
cast=[( + (1)*y) * ( + (1)*x + (A))]
other=[ + (1)*x*y + (B)*y*z]
```

3.4 Subtraction

In the subtraction, we want to perform those integrations that lead to ϵ divergencies. The master formula for one integration variables is

$$\int_0^1 x^{(a-b\epsilon)}\mathcal{I}(x,\epsilon)dx = \sum_{p=0}^{|a|-1} \frac{1}{a+p+1-b\epsilon}\frac{\mathcal{I}^{(p)}(0,\epsilon)}{p!} + \int_0^1 x^{(a-b\epsilon)} R(x,\epsilon)dx$$

where $\mathcal{I}^{(p)}$ is denotes the p-th derivative of $\mathcal{I}$ with respect to x. The equation above effectively defines the remainder term R. All terms on the right hand side of the equation above are constructed to be free of divergencies. For more details and the generalization to multiple variables, we refer the reader to [Hei08]. In the following, we show how to use the implementation in *pySecDec*.

To initialize the subtraction, we first define a factorized expression of the form $x^{(-1-b_x\epsilon)}y^{(-2-b_y\epsilon)}\mathcal{I}(x,y,\epsilon)$:

```
>>> from pySecDec.algebra import Expression
>>> symbols = ['x','y','eps']
>>> x_monomial = Expression('x**(-1 - b_x*eps)', symbols)
>>> y_monomial = Expression('y**(-2 - b_y*eps)', symbols)
>>> cal_I = Expression('cal_I(x, y, eps)', symbols)
```

We must pack the monomials into a `pySecDec.algebra.Product`:

```
>>> from pySecDec.algebra import Product
>>> monomials = Product(x_monomial, y_monomial)
```

Although this seems to be to complete input according to the equation above, we are still missing a structure to store poles in. The function `pySecDec.subtraction.integrate_pole_part()` is designed to return an iterable of the same type as the input. That is particularly important since the output of the subtraction of one variable is the input for the subtraction of the next variable. We will see this iteration later. Initially, we do not have poles yet, therefore we define a *one* of the required type:

```
>>> from pySecDec.algebra import Pow
>>> import numpy as np
>>> polynomial_one = Polynomial(np.zeros([1,len(symbols)], dtype=int), np.array([1]),
↪symbols, copy=False)
>>> pole_part_initializer = Pow(polynomial_one, -polynomial_one)
```

`pole_part_initializer` is of type `pySecDec.algebra.Pow` and has `-polynomial_one` in the exponent. We initialize the *base* with `polynomial_one`; i.e. a one packed into a polynomial. The function `pySecDec.subtraction.integrate_pole_part()` populates the *base* with factors of $b\epsilon$ when poles arise.

We are now ready to build the `subtraction_initializer` - the `pySecDec.algebra.Product` to be passed into `pySecDec.subtraction.integrate_pole_part()`.

```
>>> from pySecDec.subtraction import integrate_pole_part
>>> subtraction_initializer = Product(monomials, pole_part_initializer, cal_I)
>>> x_subtracted = integrate_pole_part(subtraction_initializer, 0)
```

The second argument of *pySecDec.subtraction.integrate_pole_part()* specifies to which variable we want to apply the master formula, here we choose x. First, remember that the x monomial is a dimensionally regulated x^-1. Therefore, the sum collapses to only one term and we have two terms in total. Each term corresponds to one entry in the list `x_subtracted`:

```
>>> len(x_subtracted)
2
```

`x_subtracted` has the same structure as our input. The first factor of each term stores the remaining monomials:

```
>>> x_subtracted[0].factors[0]
(( + (1))**( + (-b_x)*eps + (-1))) * (( + (1)*y)**( + (-b_y)*eps + (-2)))
>>> x_subtracted[1].factors[0]
(( + (1)*x)**( + (-b_x)*eps + (-1))) * (( + (1)*y)**( + (-b_y)*eps + (-2)))
```

The second factor stores the ϵ poles. There is an epsilon pole in the first term, but still none in the second:

```
>>> x_subtracted[0].factors[1]
( + (-b_x)*eps) ** ( + (-1))
>>> x_subtracted[1].factors[1]
( + (1)) ** ( + (-1))
```

The last factor catches everything that is not covered by the first two fields:

```
>>> x_subtracted[0].factors[2]
(cal_I( + (0), + (1)*y, + (1)*eps))
>>> x_subtracted[1].factors[2]
(cal_I( + (1)*x, + (1)*y, + (1)*eps)) + (( + (-1)) * (cal_I( + (0), + (1)*y, +␣
↪(1)*eps)))
```

We have now performed the subtraction for x. Because in and output have a similar structure, we can easily perform the subtraction for y as well:

```
>>> x_and_y_subtracted = []
>>> for s in x_subtracted:
...     x_and_y_subtracted.extend( integrate_pole_part(s,1) )
```

Alternatively, we can directly instruct *pySecDec.subtraction.integrate_pole_part()* to perform both subtractions:

```
>>> alternative_x_and_y_subtracted = integrate_pole_part(subtraction_initializer,0,1)
```

In both cases, the result is a list of the terms appearing on the right hand side of the master equation.

3.5 Expansion

The purpose of the *expansion* module is, as the name suggests, to provide routines to perform a series expansion. The module basically implements two routines - the Taylor expansion (*pySecDec.expansion.expand_Taylor()*) and an expansion of polyrational functions supporting singularities in the expansion variable (*pySecDec.expansion.expand_singular()*).

3.5.1 Taylor Expansion

The function *pySecDec.expansion.expand_Taylor()* implements the ordinary Taylor expansion. It takes an algebraic expression (in the sense of the *algebra module*, the index of the expansion variable and the order to which the expression shall be expanded:

```
>>> from pySecDec.algebra import Expression
>>> from pySecDec.expansion import expand_Taylor
>>> expression = Expression('x**eps', ['eps'])
>>> expand_Taylor(expression, 0, 2).simplify()
 + (1) + (log( + (x)))*eps + ((log( + (x))) * (log( + (x))) * ( + (1/2)))*eps**2
```

It is also possible to expand an expression in multiple variables simultaneously:

```
>>> expression = Expression('x**(eps + alpha)', ['eps', 'alpha'])
>>> expand_Taylor(expression, [0,1], [2,0]).simplify()
 + (1) + (log( + (x)))*eps + ((log( + (x))) * (log( + (x))) * ( + (1/2)))*eps**2
```

The command above instructs `pySecDec.expansion.expand_Taylor()` to expand the `expression` to the second order in the variable indexed `0` (`eps`) and to the zeroth order in the variable indexed `1` (`alpha`).

3.5.2 Laurent Expansion

`pySecDec.expansion.expand_singular()` Laurent expands polyrational functions.

Its input is more restrictive than for the *Taylor expansion*. It expects a `Product` where the factors are either `Polynomials` or `ExponentiatedPolynomials` with `exponent = -1`:

```
>>> from pySecDec.expansion import expand_singular
>>> expression = Expression('1/(eps + alpha)', ['eps', 'alpha']).simplify()
>>> expand_singular(expression, 0, 1)
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
  File "/home/pcl340a/sjahn/Projects/pySecDec/pySecDec/expansion.py", line 241, in␣
↪expand_singular
    return _expand_and_flatten(product, indices, orders, _expand_singular_step)
  File "/home/pcl340a/sjahn/Projects/pySecDec/pySecDec/expansion.py", line 209, in _
↪expand_and_flatten
    expansion = recursive_expansion(expression, indices, orders)
  File "/home/pcl340a/sjahn/Projects/pySecDec/pySecDec/expansion.py", line 198, in␣
↪recursive_expansion
    expansion = expansion_one_variable(expression, index, order)
  File "/home/pcl340a/sjahn/Projects/pySecDec/pySecDec/expansion.py", line 82, in _
↪expand_singular_step
    raise TypeError('`product` must be a `Product`')
TypeError: `product` must be a `Product`
>>> expression # ``expression`` is indeed a polyrational function.
( + (1)*alpha + (1)*eps)**(-1)
>>> type(expression) # It is just not packed in a ``Product`` as ``expand_singular``␣
↪expects.
<class 'pySecDec.algebra.ExponentiatedPolynomial'>
>>> from pySecDec.algebra import Product
>>> expression = Product(expression)
>>> expand_singular(expression, 0, 1)
 + (( + (1)) * (( + (1)*alpha)**(-1))) + (( + (-1)) * (( + (1)*alpha**2)**(-1)))*eps
```

Like in the *Taylor expansion*, we can expand simultaneously in multiple parameters. Note, however, that the result of the Laurent expansion depends on the ordering of the expansion variables. The second argument of `pySecDec.expansion.expand_singular()` determines the order of the expansion:

```
>>> expression = Expression('1/(2*eps) * 1/(eps + alpha)', ['eps', 'alpha']).
↪simplify()
>>> eps_first = expand_singular(expression, [0,1], [1,1])
>>> eps_first
 + (( + (1/2)) * (( + (1))**(-1)))*eps**-1*alpha**-1 + (( + (-1/2)) * (( + (1))**(-
↪1)))*alpha**-2 + (( + (1)) * (( + (2))**(-1)))*eps*alpha**-3
>>> alpha_first = expand_singular(expression, [1,0], [1,1])
>>> alpha_first
 + (( + (1/2)) * (( + (1))**(-1)))*eps**-2 + (( + (-1/2)) * (( + (1))**(-1)))*eps**-
↪3*alpha
```

The expression printed out by our algebra module are quite messy. In order to obtain nicer output, we can convert these expressions to the slower but more high level *sympy*:

```
>>> import sympy as sp
>>> eps_first = expand_singular(expression, [0,1], [1,1])
>>> alpha_first = expand_singular(expression, [1,0], [1,1])
>>> sp.sympify(eps_first)
1/(2*alpha*eps) - 1/(2*alpha**2) + eps/(2*alpha**3)
>>> sp.sympify(alpha_first)
-alpha/(2*eps**3) + 1/(2*eps**2)
```

CHAPTER 4

SecDecUtil

SecDecUtil is a standalone autotools-c++ package, that collects common helper classes and functions needed by the c++ code generated using `loop_package` or `make_package`. Everything defined by the *SecDecUtil* is put into the c++ namepace *secdecutil*.

4.1 Series

A class template for containing (optionally truncated) Laurent series. Multivariate series can be represented as series of series.

This class overloads the arithmetic operators (`+`, `-`, `*`, `/`) and the comparator operators (`==`, `!=`). A string representation can be obtained using the `<<` operator. The `at(i)` and `[i]` operators return the coefficient of the `i`$^{\text{th}}$ power of the expansion parameter. Otherwise elements can be accessed identically to `std::vector`.

template<typename **T**>
class Series

std::string **expansion_parameter**
A string representing the expansion parameter of the series (default `x`)

int **get_order_min**() **const**
Returns the lowest order in the series.

int **get_order_max**() **const**
Returns the highest order in the series.

bool **get_truncated_above**() **const**
Checks whether the series is truncated from above.

bool **has_term**(int *order*) **const**
Checks whether the series has a term at order `order`.

Series(int *order_min*, int *order_max*, std::vector<*T*> *content*, bool *truncated_above* = true, **const** std::string *expansion_parameter* = "x")

Example:

```
#include <iostream>
#include <secdecutil/series.hpp>

int main()
{
    secdecutil::Series<int> exact(-2,1,{1,2,3,4},false,"eps");
    secdecutil::Series<int> truncated(-2,1,{1,2,3,4},true,"eps");
    secdecutil::Series<secdecutil::Series<int>> multivariate(1,2,
                                                              {
                                                                  {-2,-1,{1,2},false,
↪"alpha"},
                                                                  {-2,-1,{3,4},false,
↪"alpha"},
                                                              },false,"eps"
                                                              );

    std::cout << "exact:        " << exact << std::endl;
    std::cout << "truncated:    " << truncated << std::endl;
    std::cout << "multivariate: " << multivariate << std::endl << std::endl;

    std::cout << "exact + 1:         " << exact + 1 << std::endl;
    std::cout << "exact * exact:     " << exact * exact << std::endl;
    std::cout << "exact * truncated: " << exact * truncated << std::endl;
    std::cout << "exact.at(-2):      " << exact.at(-2) << std::endl;
}
```

Compile/Run:

```
$ c++ -I${SECDEC_CONTRIB}/include -std=c++11 example.cpp -o example -lm && ./example
```

Output:

```
exact:         + (1)*eps^-2 + (2)*eps^-1 + (3) + (4)*eps
truncated:     + (1)*eps^-2 + (2)*eps^-1 + (3) + (4)*eps + O(eps^2)
multivariate:  + ( + (1)*alpha^-2 + (2)*alpha^-1)*eps + ( + (3)*alpha^-2 + (4)*alpha^-
↪1)*eps^2

exact + 1:          + (1)*eps^-2 + (2)*eps^-1 + (4) + (4)*eps
exact * exact:      + (1)*eps^-4 + (4)*eps^-3 + (10)*eps^-2 + (20)*eps^-1 + (25) +␣
↪(24)*eps + (16)*eps^2
exact * truncated:  + (1)*eps^-4 + (4)*eps^-3 + (10)*eps^-2 + (20)*eps^-1 + O(eps^0)
exact.at(-2):       1
```

4.2 Deep Apply

A general concept to apply a `std::function` to a nested data structure. If the applied `std::function` is not void then `deep_apply()` returns a nested data structure of the return values. Currently *secdecutil* implements this for `std::vector` and `Series`.

This concept allows, for example, the elements of a nested series to be edited without knowing the depth of the nested structure.

template<typename **Tout**, typename **Tin**, template<typename...> class **Tnest**>
Tnest<*Tout*> **deep_apply** (*Tnest*<*Tin*> &*nest*, std::function<*Tout*) *Tin*
> &*func*

Example (complex conjugate a *Series*):

```
#include <iostream>
#include <complex>
#include <secdecutil/series.hpp>
#include <secdecutil/deep_apply.hpp>

int main()
{
    std::function<std::complex<double>(std::complex<double>)> conjugate =
    [] (std::complex<double> element)
    {
        return std::conj(element);
    };

    secdecutil::Series<std::complex<double>> u(-1,0,{{1,2},{3,4}},false,"eps");
    secdecutil::Series<secdecutil::Series<std::complex<double>>> m(1,1,{{1,1,{{1,2}},
↪false,"alpha"},},false,"eps");

    std::cout << "u: " << u << std::endl;
    std::cout << "m: " << m << std::endl << std::endl;

    std::cout << "conjugated u:   " << secdecutil::deep_apply(u, conjugate) <<␣
↪std::endl;
    std::cout << "conjugated m: " << secdecutil::deep_apply(m, conjugate) <<␣
↪std::endl;
}
```

Compile/Run:

```
$ c++ -I${SECDEC_CONTRIB}/include -std=c++11 example.cpp -o example -lm && ./example
```

Output:

```
u:  + ((1,2))*eps^-1 + ((3,4))
m:  + ( + ((1,2))*alpha)*eps

conjugated u:    + ((1,-2))*eps^-1 + ((3,-4))
conjugated m:  + ( + ((1,-2))*alpha)*eps
```

4.3 Uncertainties

A class template which implements uncertainty propagation for uncorrelated random variables by overloads of the +, −, * and partially /. Division by *UncorrelatedDeviation* is not implemented as it is not always defined. It has special overloads for `std::complex<T>`.

Note: Division by *UncorrelatedDeviation* is not implemented as this operation is not always well defined. Specifically, it is ill defined in the case that the errors are Gaussian distributed as the expectation value,

$$\mathrm{E}\left[\frac{1}{X}\right] = \int_{-\infty}^{\infty} \frac{1}{X} p(X)\,\mathrm{d}X,$$

where

$$p(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

is undefined in the Riemann or Lebesgue sense. The rule $\delta(a/b) = |a/b|\sqrt{(\delta a/a)^2 + (\delta b/b)^2}$ can not be derived from the first principles of probability theory.

The rules implemented for real valued error propagation are:

$$\delta(a+b) = \sqrt{(\delta a)^2 + (\delta b)^2},$$

$$\delta(a-b) = \sqrt{(\delta a)^2 + (\delta b)^2},$$

$$\delta(ab) = \sqrt{(\delta a)^2 b^2 + (\delta b)^2 a^2 + (\delta a)^2 (\delta b)^2}.$$

For complex numbers the above rules are implemented for the real and imaginary parts individually.

template<typename **T**>
class UncorrelatedDeviation

T **value**
The expectation value.

T **uncertainty**
The standard deviation.

Example:

```
#include <iostream>
#include <complex>
#include <secdecutil/uncertainties.hpp>

int main()
{
    secdecutil::UncorrelatedDeviation<double> r(1.,0.5);
    secdecutil::UncorrelatedDeviation<std::complex<double>> c({2.,3.},{0.6,0.7});

    std::cout << "r: " << r << std::endl;
    std::cout << "c: " << c << std::endl << std::endl;

    std::cout << "r.value:       " << r.value << std::endl;
    std::cout << "r.uncertainty: " << r.uncertainty << std::endl;
    std::cout << "r + c:         " << r + c << std::endl;
    std::cout << "r * c:         " << r * c << std::endl;
    std::cout << "r / 3.0:       " << r / 3. << std::endl;
    // std::cout << "1. / r:      " << 1. / r << std::endl; // ERROR
    // std::cout << "c / r:       " << c / r << std::endl;  // ERROR
}
```

Compile/Run:

```
$ c++ -I${SECDEC_CONTRIB}/include -std=c++11 example.cpp -o example -lm && ./example
```

Output:

```
r: 1 +/- 0.5
c: (2,3) +/- (0.6,0.7)

r.value:       1
r.uncertainty: 0.5
r + c:         (3,3) +/- (0.781025,0.7)
r * c:         (2,3) +/- (1.20416,1.69189)
r / 3.0:       0.333333 +/- 0.166667
```

4.4 Integrand Container

A class template for containing integrands. It stores the number of integration variables and the integrand as a `std::function`.

This class overloads the arithmetic operators (`+`, `-`, `*`, `/`) and the call operator (`()`).

template<typename **T**, typename ...**Args**>
class IntegrandContainer

int **number_of_integration_variables**
The number of integration variables that the integrand depends on.

std::function<*T* (*Args*...)> **integrand**
The integrand function. The call operator forwards to this function.

Example (add two *IntegrandContainer* and evaluate one point):

```
#include <iostream>
#include <secdecutil/integrand_container.hpp>

int main()
{
    using input_t = const double * const;
    using return_t = double;

    std::function<return_t(input_t)> f1 = [] (input_t x) { return 2*x[0]; };
    secdecutil::IntegrandContainer<return_t,input_t> c1(1,f1);

    std::function<return_t(input_t)> f2 = [] (input_t x) { return x[0]*x[1]; };
    secdecutil::IntegrandContainer<return_t,input_t> c2(2,f2);

    secdecutil::IntegrandContainer<return_t,input_t> c3 = c1 + c2;
    const double point[]{1.0,2.0};

    std::cout << "c1.number_of_integration_variables: " << c1.number_of_integration_
↪variables << std::endl;
    std::cout << "c2.number_of_integration_variables: " << c2.number_of_integration_
↪variables << std::endl << std::endl;

    std::cout << "c3.number_of_integration_variables: " << c3.number_of_integration_
↪variables << std::endl;
    std::cout << "c3.integrand(point):                " << c3.integrand(point) <<␣
↪std::endl;
}
```

Compile/Run:

```
$ c++ -I${SECDEC_CONTRIB}/include -std=c++11 example.cpp -o example -lm && ./example
```

Output:

```
c1.number_of_integration_variables: 1
c2.number_of_integration_variables: 2

c3.number_of_integration_variables: 2
c3.integrand(point):                4
```

4.5 Integrator

A base class template from which integrator implementations inherit. It defines the minimal API available for all integrators.

template<typename **return_t**, typename **input_t**, typename **container_t** = secdecutil::*IntegrandContainer*<*return_t*, *in*
class Integrator

bool **together**

(Only available if `return_t` is a `std::complex` type) If `true` after each call of the function both the real and imaginary parts are passed to the underlying integrator. If `false` after each call of the function only the real or imaginary part is passed to the underlying integrator. For some adaptive integrators considering the real and imaginary part of a complex function separately can improve the sampling. Default: `false`.

UncorrelatedDeviation<*return_t*> **integrate**(**const** *IntegrandContainer*<*return_t*, *input_t* **const** ***const** >&)

Integrates the `IntegrandContainer` and returns the value and uncertainty as an `UncorrelatedDeviation`.

An integrator that chooses another integrator based on the dimension of the integrand.

template<typename **return_t**, typename **input_t**>
class MultiIntegrator

Integrator<*return_t*, *input_t*> &**low_dimensional_integrator**

Reference to the integrator to be used if the integrand has a lower dimension than `critical_dim`.

Integrator<*return_t*, *input_t*> &**high_dimensional_integrator**

Reference to the integrator to be used if the integrand has dimension `critical_dim` or higher.

int **critical_dim**

The dimension below which the `low_dimensional_integrator` is used.

4.5.1 CQuad

For one dimensional integrals, we wrap the cquad integrator form the GNU scientifc library (gsl).

CQuad takes the following options:

- `epsrel` - The desired relative accuracy for the numerical evaluation. Default: `0.01`.
- `epsabs` - The desired absolute accuracy for the numerical evaluation. Default: `1e-7`.
- `n` - The size of the workspace. This value can only be set in the constructor. Changing this attribute of an instance is not possible. Default: `100`.
- `verbose` - Whether or not to print status information. Default: `false`.
- `zero_border` - The minimal value an integration variable can take. Default: `0.0`. (*new in version 1.3*)

4.5.2 Qmc

The quasi-monte carlo integrator as described in [PSD18]. Using a quasi-monte integrator to compute sector decomposed integrals was pioneered in [LWY+15].

template<typename **return_t**, integrators::U **maxdim**, template<typename, typename, integrators::U> class **transform_t**, typena
class Qmc : *Integrator*<*return_t*, *return_t*, *container_t*>, **public** integrators::*Qmc*<*return_t*, *return_t*, *maxdim*, *transform_t*, *fitfunc*

Derived from `secdecutil::Integrator` and `::integrators::Qmc` - the underlying standalone implementation of the Qmc.

The most important fields and template argments of `Qmc` are:

- `minn` - The minimal number of points in the Qmc lattice. Will be augmented to the next larger available `n`.
- `minm` - The minimal number of random shifts.
- `maxeval` - The maximal number of integrand evaluations.
- `epsrel` - The desired relative accuracy for the numerical evaluation.
- `epsabs` - The desired absolute accuracy for the numerical evaluation.
- `maxdim` - The highest dimension the `Qmc` instance can be used for.
- `transform_t` - The periodizing transform to apply prior to integration.
- `fitfunction_t` - The fit function transform to apply for adaptive integration.
- `verbosity` - Controls the amount of status messages during integration. Can be `0`, `1`, `2`, or `3`.
- `devices` - A `std::set` of devices to run on. `-1` denotes the CPU, positive integers refer to GPUs.

Refer to the documentation of the standalone Qmc for the default values and additional information.

An integral transform has to be chosen by setting the template argument `transform_t`. Available transforms are e.g. `Korobov<r0,r1>` and `Sidi<r0>`, please refer to the underlying Qmc implementation for a complete list. The fit function for adaptive integration can be set by the `fitfunction_t`, e.g. `PolySingular`. If not set, the default of the underlying Qmc implementation is used.

Examples how to use the Qmc *on the CPU* and on *both, CPU and GPU* are shown below.

4.5.3 Cuba

Currently we wrap the following Cuba integrators:

- `Vegas`
- `Suave`
- `Divonne`
- `Cuhre`

The Cuba integrators all implement:

- `epsrel` - The desired relative accuracy for the numerical evaluation. Default: `0.01`.
- `epsabs` - The desired absolute accuracy for the numerical evaluation. Default: `1e-7`.
- `flags` - Sets the Cuba verbosity flags. The `flags=2` means that the Cuba input parameters and the result after each iteration are written to the log file of the numerical integration. Default: `0`.
- `seed` - The seed used to generate random numbers for the numerical integration with Cuba. Default: `0`.
- `mineval` - The number of evaluations which should at least be done before the numerical integrator returns a result. Default: `0`.
- `maxeval` - The maximal number of evaluations to be performed by the numerical integrator. Default: `1000000`.

- `zero_border` - The minimal value an integration variable can take. Default: `0.0`. (*new in version 1.3*)

The available integrator specific parameters and their default values are:

Vegas	Suave	Divonne	Cuhre
nstart (`10000`)	nnew (`1000`)	key1 (`2000`)	key (`0`)
nincrease (`5000`)	nmin (`10`)	key2 (`1`)	
nbatch (`500`)	flatness (`25.0`)	key3 (`1`)	
		maxpass (`4`)	
		border (`0.0`)	
		maxchisq (`1.0`)	
		mindeviation (`0.15`)	

For the description of these more specific parameters we refer to the Cuba manual.

4.5.4 Examples

Integrate Real Function with Cuba Vegas

Example:

```
#include <iostream>
#include <secdecutil/integrand_container.hpp>
#include <secdecutil/uncertainties.hpp>
#include <secdecutil/integrators/cuba.hpp>

int main()
{
    using input_t = const double * const;
    using return_t = double;

    secdecutil::cuba::Vegas<return_t> integrator;
    integrator.epsrel = 1e-4;
    integrator.maxeval = 1e7;

    secdecutil::IntegrandContainer<return_t,input_t> c(2, [] (input_t x) { return␣
↪x[0]*x[1]; });
    secdecutil::UncorrelatedDeviation<return_t> result = integrator.integrate(c);

    std::cout << "result: " << result << std::endl;
}
```

Compile/Run:

```
$ c++ -I${SECDEC_CONTRIB}/include -L${SECDEC_CONTRIB}/lib -std=c++11 example.cpp -o␣
↪example -lcuba -lm && ./example
```

Output:

```
result: 0.250004 +/- 2.43875e-05
```

Integrate Complex Function with Cuba Vegas

Example:

```
#include <iostream>
#include <complex>
#include <secdecutil/integrand_container.hpp>
#include <secdecutil/uncertainties.hpp>
#include <secdecutil/integrators/cuba.hpp>

int main()
{
    using input_t = const double * const;
    using return_t = std::complex<double>;

    secdecutil::cuba::Vegas<return_t> integrator;
    std::function<return_t(input_t)> f = [] (input_t x) { return return_t{x[0],x[1]};
↪};
    secdecutil::IntegrandContainer<return_t,input_t> c(2,f);

    integrator.together = false; // integrate real and imaginary part separately
↪(default)
    secdecutil::UncorrelatedDeviation<return_t> result_separate = integrator.
↪integrate(c);

    integrator.together = true; // integrate real and imaginary part simultaneously
    secdecutil::UncorrelatedDeviation<return_t> result_together = integrator.
↪integrate(c);

    std::cout << "result_separate: " << result_separate << std::endl;
    std::cout << "result_together: " << result_together << std::endl;

}
```

Compile/Run:

```
$ c++ -I${SECDEC_CONTRIB}/include -L${SECDEC_CONTRIB}/lib -std=c++11 example.cpp -o
↪example -lcuba -lm && ./example
```

Output:

```
result_separate: (0.499937,0.499937) +/- (0.00288675,0.00288648)
result_together: (0.499937,0.499937) +/- (0.00288675,0.00288648)
```

Integrate Real Function with Cuba Vegas or CQuad

Example:

```
#include <iostream>
#include <secdecutil/integrand_container.hpp>
#include <secdecutil/uncertainties.hpp>
#include <secdecutil/integrators/integrator.hpp>
#include <secdecutil/integrators/cuba.hpp>
#include <secdecutil/integrators/cquad.hpp>

int main()
{
    using input_base_t = double;
    using input_t = const input_base_t * const;
    using return_t = double;
```

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```
    secdecutil::cuba::Vegas<return_t> vegas;
    vegas.epsrel = 1e-5;
    vegas.maxeval = 1e7;

    secdecutil::gsl::CQuad<return_t> cquad;
    cquad.epsrel = 1e-10;
    cquad.epsabs = 1e-13;

    secdecutil::MultiIntegrator<return_t,input_base_t> integrator(cquad,vegas,2);

    secdecutil::IntegrandContainer<return_t,input_t> one_dimensional(1, [] (input_t␣
↪x) { return x[0]; });
    secdecutil::IntegrandContainer<return_t,input_t> two_dimensional(2, [] (input_t␣
↪x) { return x[0]*x[1]; });

    secdecutil::UncorrelatedDeviation<return_t> result_1d = integrator.integrate(one_
↪dimensional); // uses cquad
    secdecutil::UncorrelatedDeviation<return_t> result_2d = integrator.integrate(two_
↪dimensional); // uses vegas

    std::cout << "result_1d: " << result_1d << std::endl;
    std::cout << "result_2d: " << result_2d << std::endl;
}
```

Compile/Run:

```
$ c++ -I${SECDEC_CONTRIB}/include -L${SECDEC_CONTRIB}/lib -std=c++11 example.cpp -o␣
↪example -lcuba -lgsl -lgslcblas -lm && ./example
```

Output:

```
result_1d: 0.5 +/- 9.58209e-17
result_2d: 0.25 +/- 5.28257e-06
```

Set the integral transform of the Qmc

Example:

```
#include <iostream>
#include <secdecutil/integrand_container.hpp>
#include <secdecutil/uncertainties.hpp>
#include <secdecutil/integrators/qmc.hpp>

using input_base_t = double;
using input_t = input_base_t const * const;
using return_t = double;
using container_t = secdecutil::IntegrandContainer<return_t,input_t>;
using result_t = secdecutil::UncorrelatedDeviation<return_t>;

const int seed = 12345, maxdim = 4;

int main()
{
    /*
```

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```
     * minimal instantiation
     */
    secdecutil::integrators::Qmc
    <
        return_t, // the return type of the integrand
        maxdim, // the highest dimension this integrator will be used for
        ::integrators::transforms::Baker::type // the integral transform
    > integrator_baker;
    integrator_baker.randomgenerator.seed(seed);

    /*
     * disable adaptation
     */
    secdecutil::integrators::Qmc
    <
        return_t, // the return type of the integrand
        maxdim, // the highest dimension this integrator will be used for
        ::integrators::transforms::Korobov<4,1>::type, // the integral transform
        container_t, // the functor type to be passed to this integrator
        ::integrators::fitfunctions::None::type // the fit funtion
    > integrator_korobov4x1;
    integrator_korobov4x1.randomgenerator.seed(seed);

    /*
     * enable adaptation
     */
    secdecutil::integrators::Qmc
    <
        return_t, // the return type of the integrand
        maxdim, // the highest dimension this integrator will be used for
        ::integrators::transforms::Sidi<3>::type, // the integral transform
        container_t, // the functor type to be passed to this integrator
        ::integrators::fitfunctions::PolySingular::type // the fit funtion
    > integrator_sidi3_adaptive;
    integrator_sidi3_adaptive.randomgenerator.seed(seed);

    // define the integrand as a functor
    container_t integrand(
                                 4, // dimension
                                 [] (input_t x) { return x[0]*x[1]*x[2]*x[3]; } //␣
↪integrand function
                             );

    // compute the integral with different settings
    result_t result_baker = integrator_baker.integrate(integrand);
    result_t result_korobov4x1 = integrator_korobov4x1.integrate(integrand);
    result_t result_sidi3_adaptive = integrator_sidi3_adaptive.integrate(integrand);

    // print the results
    std::cout << "baker: " << result_baker << std::endl;
    std::cout << "Korobov (weights 4, 1): " << result_korobov4x1 << std::endl;
    std::cout << "Sidi (weight 3, adaptive): " << result_sidi3_adaptive << std::endl;
}
```

Compile/Run:

```
c++ -I${SECDEC_CONTRIB}/include -pthread -L${SECDEC_CONTRIB}/lib -std=c++11 example.
↪cpp -o example -lm && ./example
```

Output:

```
baker: 0.0625 +/- 7.93855e-08
Korobov (weights 4, 1): 0.0625108 +/- 2.97931e-05
Sidi (weight 3, adaptive): 0.0625 +/- 4.33953e-09
```

Run the Qmc on GPUs

Example:

```
#include <iostream>
#include <secdecutil/integrand_container.hpp>
#include <secdecutil/uncertainties.hpp>
#include <secdecutil/integrators/qmc.hpp>

using input_base_t = double;
using input_t = input_base_t const * const;
using return_t = double;
using container_t = secdecutil::IntegrandContainer<return_t,input_t>;
using result_t = secdecutil::UncorrelatedDeviation<return_t>;

/*
 * `container_t` cannot be used on the GPU --> define a different container type
 */
struct cuda_integrand_t
{
    const int number_of_integration_variables = 4;

    // integrand function
    #ifdef __CUDACC__
        __host__ __device__
    #endif
    return_t operator()(input_t x)
    {
        return x[0]*x[1]*x[2]*x[3];
    };
} cuda_integrand;

const int seed = 12345, maxdim = 4;

int main()
{
    /*
     * Qmc capable of sampling on the GPU
     */
    secdecutil::integrators::Qmc
    <
        return_t, // the return type of the integrand
        maxdim, // the highest dimension this integrator will be used for
        ::integrators::transforms::Sidi<3>::type, // the integral transform
        cuda_integrand_t, // the functor type to be passed to this integrator
        ::integrators::fitfunctions::PolySingular::type // the fit funtion (optional)
    > integrator_sidi3_adaptive_gpu;
```

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```
    integrator_sidi3_adaptive_gpu.randomgenerator.seed(seed);

    // compute the integral with different settings
    result_t result_sidi3_adaptive_gpu = integrator_sidi3_adaptive_gpu.integrate(cuda_
↪integrand);

    // print the results
    std::cout << "Sidi (weight 3, adaptive): " << result_sidi3_adaptive_gpu <<␣
↪std::endl;
}
```

Compile/Run:

```
nvcc -x cu -I${SECDEC_CONTRIB}/include -L${SECDEC_CONTRIB}/lib -std=c++11 example.cpp␣
↪-o example -lgsl -lgslcblas -lm && ./example # with GPU
c++ -I${SECDEC_CONTRIB}/include -pthread -L${SECDEC_CONTRIB}/lib -std=c++11 example.
↪cpp -o example -lgsl -lgslcblas -lm && ./example # without GPU
```

Output:

```
Sidi (weight 3, adaptive): 0.0625 +/- 4.33953e-09
```

CHAPTER 5

Reference Guide

This section describes all public functions and classes in *pySecDec*.

5.1 Algebra

Implementation of a simple computer algebra system.

class pySecDec.algebra.**ExponentiatedPolynomial**(*expolist*, *coeffs*, *exponent=1*, *polysymbols='x'*, *copy=True*)

Like `Polynomial`, but with a global exponent. $polynomial^{exponent}$

Parameters

- **expolist** – iterable of iterables; The variable's powers for each term.
- **coeffs** – iterable; The coefficients of the polynomial.
- **exponent** – object, optional; The global exponent.
- **polysymbols** – iterable or string, optional; The symbols to be used for the polynomial variables when converted to string. If a string is passed, the variables will be consecutively numbered.

  For example: expolist=[[2,0],[1,1]] coeffs=["A","B"]

  – polysymbols='x' (default) <-> "A*x0**2 + B*x0*x1"

  – polysymbols=['x','y'] <-> "A*x**2 + B*x*y"
- **copy** – bool; Whether or not to copy the *expolist*, the *coeffs*, and the *exponent*.

Note: If copy is `False`, it is assumed that the *expolist*, the *coeffs* and the *exponent* have the correct type.

copy()

Return a copy of a `Polynomial` or a subclass.

derive (*index*)
Generate the derivative by the parameter indexed *index*.

Parameters **index** – integer; The index of the paramater to derive by.

simplify ()
Apply the identity <something>**0 = 1 or <something>**1 = <something> or 1**<something> = 1 if possible, otherwise call the simplify method of the base class. Convert `exponent` to symbol if possible.

`pySecDec.algebra.`**Expression** (*expression*, *polysymbols*, *follow_functions=False*)
Convert a sympy expression to an expression in terms of this module.

Parameters

- **expression** – string or sympy expression; The expression to be converted
- **polysymbols** – iterable of strings or sympy symbols; The symbols to be stored as `expolists` (see *Polynomial*) where possible.
- **follow_functions** – bool, optional (default = `False`); If true, return the converted expression and a list of *Function* that occur in the *expression*.

class `pySecDec.algebra.`**Function** (*symbol*, **arguments*, ***kwargs*)
Symbolic function that can take care of parameter transformations. It keeps track of all taken derivatives: When *derive()* is called, save the multiindex of the taken derivative.

The derivative multiindices are the keys in the dictionary `self.derivative_tracks`. The values are lists with two elements: Its first element is the index to derive the derivative indicated by the multiindex in the second element by, in order to abtain the derivative indicated by the key:

```
>>> from pySecDec.algebra import Polynomial, Function
>>> x = Polynomial.from_expression('x', ['x','y'])
>>> y = Polynomial.from_expression('y', ['x','y'])
>>> poly = x**2*y + y**2
>>> func = Function('f', x, y)
>>> ddfuncd0d1 = func.derive(0).derive(1)
>>> func
Function(f( + (1)*x, + (1)*y), derivative_tracks = {(1, 0): [0, (0, 0)], (1, 1):␣
↪[1, (1, 0)]})
>>> func.derivative_tracks
{(1, 0): [0, (0, 0)], (1, 1): [1, (1, 0)]}
>>> func.compute_derivatives(poly)
{(1, 0):  + (2)*x*y, (1, 1):  + (2)*x}
```

Parameters

- **symbol** – string; The symbol to be used to represent the *Function*.
- **arguments** – arbitrarily many `_Expression`; The arguments of the *Function*.
- **copy** – bool; Whether or not to copy the *arguments*.

compute_derivatives (*expression=None*)
Compute all derivatives of `expression` that are mentioned in `self.derivative_tracks`. The purpose of this function is to avoid computing the same derivatives multiple times.

Parameters **expression** – `_Expression`, optional; The expression to compute the derivatives of. If not provided, the derivatives are shown as in terms of the *function*'s derivatives `dfd<index>`.

copy ()
Return a copy of a *Function*.

derive(*index*)
Generate the derivative by the parameter indexed *index*. The derivative of a function with *symbol* `f` by some *index* is denoted as `dfd<index>`.

Parameters **index** – integer; The index of the paramater to derive by.

replace(*index*, *value*, *remove=False*)
Replace a variable in an expression by a number or a symbol. The entries in all `expolist` of the underlying *Polynomial* are set to zero. The coefficients are modified according to *value* and the powers indicated in the `expolist`.

Parameters

- **expression** – `_Expression`; The expression to replace the variable.
- **index** – integer; The index of the variable to be replaced.
- **value** – number or sympy expression; The value to insert for the chosen variable.
- **remove** – bool; Whether or not to remove the replaced parameter from the `parameters` in the *expression*.

simplify()
Simplify the arguments.

class pySecDec.algebra.**Log**(*arg*, *copy=True*)
The (natural) logarithm to base e (2.718281828459..). Store the expressions `log(arg)`.

Parameters

- **arg** – `_Expression`; The argument of the logarithm.
- **copy** – bool; Whether or not to copy the *arg*.

copy()
Return a copy of a *Log*.

derive(*index*)
Generate the derivative by the parameter indexed *index*.

Parameters **index** – integer; The index of the paramater to derive by.

replace(*index*, *value*, *remove=False*)
Replace a variable in an expression by a number or a symbol. The entries in all `expolist` of the underlying *Polynomial* are set to zero. The coefficients are modified according to *value* and the powers indicated in the `expolist`.

Parameters

- **expression** – `_Expression`; The expression to replace the variable.
- **index** – integer; The index of the variable to be replaced.
- **value** – number or sympy expression; The value to insert for the chosen variable.
- **remove** – bool; Whether or not to remove the replaced parameter from the `parameters` in the *expression*.

simplify()
Apply `log(1) = 0`.

class pySecDec.algebra.**LogOfPolynomial**(*expolist*, *coeffs*, *polysymbols='x'*, *copy=True*)
The natural logarithm of a *Polynomial*.

Parameters

- **expolist** – iterable of iterables; The variable’s powers for each term.
- **coeffs** – iterable; The coefficients of the polynomial.
- **exponent** – object, optional; The global exponent.
- **polysymbols** – iterable or string, optional; The symbols to be used for the polynomial variables when converted to string. If a string is passed, the variables will be consecutively numbered.

  For example: expolist=[[2,0],[1,1]] coeffs=[“A”,”B”]

  – polysymbols=’x’ (default) <-> “A*x0**2 + B*x0*x1”

  – polysymbols=[‘x’,’y’] <-> “A*x**2 + B*x*y”

derive (*index*)
Generate the derivative by the parameter indexed *index*.

Parameters **index** – integer; The index of the paramater to derive by.

static from_expression (*expression*, *polysymbols*)
Alternative constructor. Construct the *LogOfPolynomial* from an algebraic expression.

Parameters

- **expression** – string or sympy expression; The algebraic representation of the polynomial, e.g. “5*x1**2 + x1*x2”
- **polysymbols** – iterable of strings or sympy symbols; The symbols to be interpreted as the polynomial variables, e.g. “[‘x1’,’x2’]”.

simplify ()
Apply the identity `log(1) = 0`, otherwise call the simplify method of the base class.

class pySecDec.algebra.**Polynomial** (*expolist*, *coeffs*, *polysymbols=’x’*, *copy=True*)
Container class for polynomials. Store a polynomial as list of lists counting the powers of the variables. For example the polynomial “x1**2 + x1*x2” is stored as [[2,0],[1,1]].

Coefficients are stored in a separate list of strings, e.g. “A*x0**2 + B*x0*x1” <-> [[2,0],[1,1]] and [“A”,”B”].

Parameters

- **expolist** – iterable of iterables; The variable’s powers for each term.

  Hint: Negative powers are allowed.

- **coeffs** – 1d array-like with numerical or sympy-symbolic (see http://www.sympy.org/) content, e.g. [x,1,2] where x is a sympy symbol; The coefficients of the polynomial.
- **polysymbols** – iterable or string, optional; The symbols to be used for the polynomial variables when converted to string. If a string is passed, the variables will be consecutively numbered.

  For example: expolist=[[2,0],[1,1]] coeffs=[“A”,”B”]

  – polysymbols=’x’ (default) <-> “A*x0**2 + B*x0*x1”

  – polysymbols=[‘x’,’y’] <-> “A*x**2 + B*x*y”

- **copy** – bool; Whether or not to copy the *expolist* and the *coeffs*.

Note: If copy is `False`, it is assumed that the *expolist* and the *coeffs* have the correct type.

becomes_zero_for(*zero_params*)
Return True if the polynomial becomes zero if the parameters passed in *zero_params* are set to zero. Otherwise, return False.

Parameters **zero_params** – iterable of integers; The indices of the parameters to be checked.

copy()
Return a copy of a *Polynomial* or a subclass.

derive(*index*)
Generate the derivative by the parameter indexed *index*.

Parameters **index** – integer; The index of the paramater to derive by.

static from_expression(*expression*, *polysymbols*)
Alternative constructor. Construct the polynomial from an algebraic expression.

Parameters

- **expression** – string or sympy expression; The algebraic representation of the polynomial, e.g. "5*x1**2 + x1*x2"
- **polysymbols** – iterable of strings or sympy symbols; The symbols to be interpreted as the polynomial variables, e.g. "['x1','x2']".

has_constant_term(*indices=None*)
Return True if the polynomial can be written as:

$$const + ...$$

Otherwise, return False.

Parameters **indices** – list of integers or None; The indices of the *polysymbols* to consider. If `None` (default) all indices are taken into account.

replace(*index*, *value*, *remove=False*)
Replace a variable in an expression by a number or a symbol. The entries in all `expolist` of the underlying *Polynomial* are set to zero. The coefficients are modified according to *value* and the powers indicated in the `expolist`.

Parameters

- **expression** – `_Expression`; The expression to replace the variable.
- **index** – integer; The index of the variable to be replaced.
- **value** – number or sympy expression; The value to insert for the chosen variable.
- **remove** – bool; Whether or not to remove the replaced parameter from the `parameters` in the *expression*.

simplify(*deep=True*)
Combine terms that have the same exponents of the variables.

Parameters **deep** – bool; If `True` (default) call the *simplify* method of the coefficients if they are of type `_Expression`.

class pySecDec.algebra.**Pow**(*base*, *exponent*, *copy=True*)
Exponential. Store two expressions `A` and `B` to be interpreted as the exponential `A**B`.

Parameters

- **base** – `_Expression`; The base `A` of the exponential.
- **exponent** – `_Expression`; The exponent `B`.
- **copy** – bool; Whether or not to copy *base* and *exponent*.

copy()
Return a copy of a *Pow*.

derive(*index*)
Generate the derivative by the parameter indexed *index*.

Parameters **index** – integer; The index of the paramater to derive by.

replace(*index*, *value*, *remove=False*)
Replace a variable in an expression by a number or a symbol. The entries in all `expolist` of the underlying *Polynomial* are set to zero. The coefficients are modified according to *value* and the powers indicated in the `expolist`.

Parameters

- **expression** – `_Expression`; The expression to replace the variable.
- **index** – integer; The index of the variable to be replaced.
- **value** – number or sympy expression; The value to insert for the chosen variable.
- **remove** – bool; Whether or not to remove the replaced parameter from the `parameters` in the *expression*.

simplify()
Apply the identity <something>**0 = 1 or <something>**1 = <something> or 1**<something> = 1 if possible. Convert to *ExponentiatedPolynomial* or *Polynomial* if possible.

class pySecDec.algebra.**Product**(**factors*, ***kwargs*)
Product of polynomials. Store one or polynomials p_i to be interpreted as product $\prod_i p_i$.

Parameters

- **factors** – arbitrarily many instances of *Polynomial*; The factors p_i.
- **copy** – bool; Whether or not to copy the *factors*.

p_i can be accessed with `self.factors[i]`.

Example:

```
p = Product(p0, p1)
p0 = p.factors[0]
p1 = p.factors[1]
```

copy()
Return a copy of a *Product*.

derive(*index*)
Generate the derivative by the parameter indexed *index*. Return an instance of the optimized *ProductRule*.

Parameters **index** – integer; The index of the paramater to derive by.

replace(*index*, *value*, *remove=False*)
Replace a variable in an expression by a number or a symbol. The entries in all `expolist` of the underlying *Polynomial* are set to zero. The coefficients are modified according to *value* and the powers indicated in the `expolist`.

Parameters

- **expression** – _Expression; The expression to replace the variable.
- **index** – integer; The index of the variable to be replaced.
- **value** – number or sympy expression; The value to insert for the chosen variable.
- **remove** – bool; Whether or not to remove the replaced parameter from the parameters in the *expression*.

simplify()

If one or more of self.factors is a *Product*, replace it by its factors. If only one factor is present, return that factor. Remove factors of one and zero.

class pySecDec.algebra.**ProductRule**(**expressions*, ***kwargs*)

Store an expression of the form

$$\sum_i c_i \prod_j \prod_k \left(\frac{d}{dx_k} \right)^{n_{ijk}} f_j \left(\{x_k\} \right)$$

The main reason for introducing this class is a speedup when calculating derivatives. In particular, this class implements simplifications such that the number of terms grows less than exponentially (scaling of the naive implementation of the product rule) with the number of derivatives.

Parameters **expressions** – arbitrarily many expressions; The expressions f_j.

copy()

Return a copy of a *ProductRule*.

derive(*index*)

Generate the derivative by the parameter indexed *index*. Note that this class is particularly designed to hold derivatives of a product.

Parameters **index** – integer; The index of the paramater to derive by.

replace(*index*, *value*, *remove=False*)

Replace a variable in an expression by a number or a symbol. The entries in all expolist of the underlying *Polynomial* are set to zero. The coefficients are modified according to *value* and the powers indicated in the expolist.

Parameters

- **expression** – _Expression; The expression to replace the variable.
- **index** – integer; The index of the variable to be replaced.
- **value** – number or sympy expression; The value to insert for the chosen variable.
- **remove** – bool; Whether or not to remove the replaced parameter from the parameters in the *expression*.

simplify()

Combine terms that have the same derivatives of the *expressions*.

to_sum()

Convert the *ProductRule* to *Sum*

class pySecDec.algebra.**Sum**(**summands*, ***kwargs*)

Sum of polynomials. Store one or polynomials p_i to be interpreted as product $\sum_i p_i$.

Parameters

- **summands** – arbitrarily many instances of *Polynomial*; The summands p_i.

- **copy** – bool; Whether or not to copy the *summands*.

p_i can be accessed with `self.summands[i]`.

Example:

```
p = Sum(p0, p1)
p0 = p.summands[0]
p1 = p.summands[1]
```

copy()
Return a copy of a *Sum*.

derive(*index*)
Generate the derivative by the parameter indexed *index*.

Parameters **index** – integer; The index of the paramater to derive by.

replace(*index*, *value*, *remove=False*)
Replace a variable in an expression by a number or a symbol. The entries in all `expolist` of the underlying *Polynomial* are set to zero. The coefficients are modified according to *value* and the powers indicated in the `expolist`.

Parameters

- **expression** – `_Expression`; The expression to replace the variable.
- **index** – integer; The index of the variable to be replaced.
- **value** – number or sympy expression; The value to insert for the chosen variable.
- **remove** – bool; Whether or not to remove the replaced parameter from the `parameters` in the *expression*.

simplify()
If one or more of `self.summands` is a *Sum*, replace it by its summands. If only one summand is present, return that summand. Remove zero from sums.

5.2 Loop Integral

This module defines routines to Feynman parametrize a loop integral and build a c++ package that numerically integrates over the sector decomposed integrand.

5.2.1 Feynman Parametrization

Routines to Feynman parametrize a loop integral.

class pySecDec.loop_integral.**LoopIntegral**(**args*, ***kwargs*)
Container class for loop integrals. The main purpose of this class is to convert a loop integral from the momentum representation to the Feynman parameter representation.

It is possible to provide either the graph of the loop integrals as adjacency list, or the propagators.

The Feynman parametrized integral is a product of the following expressions that are accessible as member properties:

- `self.regulator ** self.regulator_power`
- `self.Gamma_factor`

- `self.exponentiated_U`
- `self.exponentiated_F`
- `self.numerator`
- `self.measure`,

where `self` is an instance of either *LoopIntegralFromGraph* or *LoopIntegralFromPropagators*.

When inverse propagators or nonnumerical propagator powers are present (see *powerlist*), some *Feynman_parameters* drop out of the integral. The variables to integrate over can be accessed as `self.integration_variables`.

While `self.numerator` describes the numerator polynomial generated by tensor numerators or inverse propagators, `self.measure` contains the monomial associated with the integration measure in the case of propagator powers $\neq 1$. The Gamma functions in the denominator belonging to the measure, however, are multiplied to the overall Gamma factor given by `self.Gamma_factor`.

Changed in version 1.2.2: The overall sign $(-1)^{N_\nu}$ is included in `self.Gamma_factor`.

See also:

- input as graph: *LoopIntegralFromGraph*
- input as list of propagators: *LoopIntegralFromPropagators*

class pySecDec.loop_integral.**LoopIntegralFromGraph**(*internal_lines, external_lines, replacement_rules=[], Feynman_parameters='x', regulator='eps', regulator_power=0, dimensionality='4-2*eps', powerlist=[]*)

Construct the Feynman parametrization of a loop integral from the graph using the cut construction method.

Example:

```
>>> from pySecDec.loop_integral import *
>>> internal_lines = [['0',[1,2]], ['m',[2,3]], ['m',[3,1]]]
>>> external_lines = [['p1',1],['p2',2],['-p12',3]]
>>> li = LoopIntegralFromGraph(internal_lines, external_lines)
>>> li.exponentiated_U
( + (1)*x0 + (1)*x1 + (1)*x2)**(2*eps - 1)
>>> li.exponentiated_F
( + (m**2)*x2**2 + (2*m**2 - p12**2)*x1*x2 + (m**2)*x1**2 + (m**2 - p1**2)*x0*x2
 + (m**2 - p2**2)*x0*x1)**(-eps - 1)
```

Parameters

- **internal_lines** – iterable of internal line specification, consisting of string or sympy expression for mass and a pair of strings or numbers for the vertices, e.g. `[['m', [1, 2]], ['0', [2,1]]]`.
- **external_lines** – iterable of external line specification, consisting of string or sympy expression for external momentum and a strings or number for the vertex, e.g. `[['p1', 1], ['p2', 2]]`.
- **replacement_rules** – iterable of iterables with two strings or sympy expressions, optional; Symbolic replacements to be made for the external momenta, e.g. definition of Mandelstam variables. Example: `[('p1*p2', 's'), ('p1**2', 0)]` where `p1` and

p2 are external momenta. It is also possible to specify vector replacements, for example `[('p4', '-(p1+p2+p3)')]`.

- **Feynman_parameters** – iterable or string, optional; The symbols to be used for the Feynman parameters. If a string is passed, the Feynman parameter variables will be consecutively numbered starting from zero.
- **regulator** – string or sympy symbol, optional; The symbol to be used for the dimensional regulator (typically ϵ or ϵ_D)

  Note: If you change this symbol, you have to adapt the *dimensionality* accordingly.

- **regulator_power** – integer; An additional factor to the *numerator*.

  See also:

  LoopIntegral

- **dimensionality** – string or sympy expression, optional; The dimensionality; typically $4 - 2\epsilon$, which is the default value.
- **powerlist** – iterable, optional; The powers of the propergators, possibly dependent on the *regulator*. In case of negative powers, the *numerator* is constructed by taking derivatives with respect to the corresponding Feynman parameters as explained in Section 3.2.4 of Ref. [BHJ+15]. If negative powers are combined with a tensor numerator, the derivatives act on the Feynman-parametrized tensor numerator as well, which leads to a consistent result.

class pySecDec.loop_integral.**LoopIntegralFromPropagators**(*propagators, loop_momenta, external_momenta=[], Lorentz_indices=[], numerator=1, metric_tensor='g', replacement_rules=[], Feynman_parameters='x', regulator='eps', regulator_power=0, dimensionality='4-2*eps', powerlist=[]*)

Construct the Feynman parametrization of a loop integral from the algebraic momentum representation.

See also:

[Hei08], [GKR+11]

Example:

```
>>> from pySecDec.loop_integral import *
>>> propagators = ['k**2', '(k - p)**2']
>>> loop_momenta = ['k']
>>> li = LoopIntegralFromPropagators(propagators, loop_momenta)
>>> li.exponentiated_U
( + (1)*x0 + (1)*x1)**(2*eps - 2)
>>> li.exponentiated_F
( + (-p**2)*x0*x1)**(-eps)
```

The 1st (U) and 2nd (F) Symanzik polynomials and their exponents can also be accessed independently:

```
>>> li.U
 + (1)*x0 + (1)*x1
>>> li.F
 + (-p**2)*x0*x1
>>>
>>> li.exponent_U
2*eps - 2
>>> li.exponent_F
-eps
```

Parameters

- **propagators** – iterable of strings or sympy expressions; The propagators, e.g. `['k1**2', '(k1-k2)**2 - m1**2']`.
- **loop_momenta** – iterable of strings or sympy expressions; The loop momenta, e.g. `['k1','k2']`.
- **external_momenta** – iterable of strings or sympy expressions, optional; The external momenta, e.g. `['p1','p2']`. Specifying the *external_momenta* is only required when a *numerator* is to be constructed.

  See also:

  parameter *numerator*
- **Lorentz_indices** – iterable of strings or sympy expressions, optional; Symbols to be used as Lorentz indices in the numerator.

  See also:

  parameter *numerator*
- **numerator** – string or sympy expression, optional; The numerator of the loop integral. Scalar products must be passed in index notation e.g. `k1(mu)*k2(mu)`. The numerator must be a sum of products of exclusively:
  - numbers
  - scalar products (e.g. `p1(mu)*k1(mu)*p1(nu)*k2(nu)`)
  - symbols (e.g. `s`, `eps`)

  Examples:

  - `p1(mu)*k1(mu)*p1(nu)*k2(nu) + 4*s*eps*k1(mu)*k1(mu)`
  - `p1(mu)*(k1(mu) + k2(mu))*p1(nu)*k2(nu)`
  - `p1(mu)*k1(mu)`

Note: In order to use the resulting *LoopIntegral* as an argument to the function `pySecDec.loop_integral.loop_package()`, the resulting Feynman parametrized `self.numerator` must be expressible as a `pySecDec.algebra.Polynomial` such that all coefficients are purely numeric. In addition, all scalar products of the external momenta must be expressed in terms of Mandelstam variables using the *replacement_rules*.

Warning: **All** Lorentz indices (including the contracted ones and also including the numbers that have been used) must be explicitly defined using the parameter *Lorentz_indices*.

Hint: It is possible to use numbers as indices, for example `p1(mu)*p2(mu)*k1(nu)*k2(nu) = p1(1)*p2(1)*k1(2)*k2(2)`.

Hint: The numerator may have uncontracted indices, e.g. `k1(mu)*k2(nu)`. If indices are left open, however, the LoopIntegral cannot be used with the package generator `pySecDec.loop_integral.loop_package()`.

- **`metric_tensor`** – string or sympy symbol, optional; The symbol to be used for the (Minkowski) metric tensor $g^{\mu\nu}$.
- **`replacement_rules`** – iterable of iterables with two strings or sympy expressions, optional; Symbolic replacements to be made for the external momenta, e.g. definition of Mandelstam variables. Example: `[('p1*p2', 's'), ('p1**2', 0)]` where `p1` and `p2` are external momenta. It is also possible to specify vector replacements, for example `[('p4', '-(p1+p2+p3)')]`.
- **`Feynman_parameters`** – iterable or string, optional; The symbols to be used for the Feynman parameters. If a string is passed, the Feynman parameter variables will be consecutively numbered starting from zero.
- **`regulator`** – string or sympy symbol, optional; The symbol to be used for the dimensional regulator (typically ϵ or ϵ_D)

  Note: If you change this symbol, you have to adapt the *dimensionality* accordingly.

- **`regulator_power`** – integer; An additional factor to the *numerator*.

  See also:

  `LoopIntegral`

- **`dimensionality`** – string or sympy expression, optional; The dimensionality; typically $4 - 2\epsilon$, which is the default value.
- **`powerlist`** – iterable, optional; The powers of the propergators, possibly dependent on the *regulator*. In case of negative powers, the *numerator* is constructed by taking derivatives with respect to the corresponding Feynman parameters as explained in Section 3.2.4 of Ref. [BHJ+15]. If negative powers are combined with a tensor numerator, the derivatives act on the Feynman-parametrized tensor numerator as well, which leads to a consistent result.

5.2.2 Loop Package

This module contains the function that generates a c++ package.

`pySecDec.loop_integral.`**`loop_package`**`(`*name, loop_integral, requested_order, real_parameters=[], complex_parameters=[], contour_deformation=True, additional_prefactor=1, form_optimization_level=2, form_work_space='500M', decomposition_method='iterative', normaliz_executable='normaliz', enforce_complex=False, split=False, ibp_power_goal=-1, use_iterative_sort=True, use_light_Pak=True, use_dreadnaut=False, use_Pak=True, processes=None*`)`

Decompose, subtract and expand a Feynman parametrized loop integral. Return it as c++ package.

See also:

This function is a wrapper around `pySecDec.code_writer.make_package()`.

See also:

The generated library is described in *Generated C++ Libraries*.

Parameters

- **`name`** – string; The name of the c++ namespace and the output directory.
- **`loop_integral`** – `pySecDec.loop_integral.LoopIntegral`; The loop integral to be computed.
- **`requested_orders`** – integer; Compute the expansion in the regulator to this order.
- **`real_parameters`** – iterable of strings or sympy symbols, optional; Parameters to be interpreted as real numbers, e.g. Mandelstam invariants and masses.
- **`complex_parameters`** – iterable of strings or sympy symbols, optional; Parameters to be interpreted as complex numbers. To use the complex mass scheme, define the masses as complex parameters.
- **`contour_deformation`** – bool, optional; Whether or not to produce code for contour deformation. Default: `True`.
- **`additional_prefactor`** – string or sympy expression, optional; An additional factor to be multiplied to the loop integral. It may depend on the regulator, the *real_parameters*, and the *complex_parameters*.
- **`form_optimization_level`** – integer out of the interval [0,4], optional; The optimization level to be used in FORM. Default: `2`.
- **`form_work_space`** – string, optional; The FORM WorkSpace. Default: `'500M'`.
- **`decomposition_method`** – string, optional; The strategy for decomposing the polynomials. The following strategies are available:
  - 'iterative' (default)
  - 'geometric'
  - 'geometric_ku'

  Note: For 'geometric' and 'geometric_ku', the third-party program "normaliz" is needed. See *The Geomethod and Normaliz*.

- **`normaliz_executable`** – string, optional; The command to run *normaliz*. *normaliz* is only required if *decomposition_method* is set to 'geometric' or 'geometric_ku'. Default: 'normaliz'

- **enforce_complex** – bool, optional; Whether or not the generated integrand functions should have a complex return type even though they might be purely real. The return type of the integrands is automatically complex if *contour_deformation* is `True` or if there are *complex_parameters*. In other cases, the calculation can typically be kept purely real. Most commonly, this flag is needed if `log(<negative real>)` occurs in one of the integrand functions. However, *pySecDec* will suggest setting this flag to `True` in that case. Default: `False`
- **split** – bool, optional; Whether or not to split the integration domain in order to map singularities from 1 to 0. Set this option to `True` if you have singularties when one or more integration variables are one. Default: `False`
- **ibp_power_goal** – number or iterable of number, optional; The *power_goal* that is forwarded to *integrate_by_parts()*.

  This option controls how the subtraction terms are generated. Setting it to `-numpy.inf` disables *integrate_by_parts()*, while `0` disables *integrate_pole_part()*.

  See also:

  To generate the subtraction terms, this function first calls *integrate_by_parts()* for each integration variable with the give *ibp_power_goal*. Then *integrate_pole_part()* is called.

  Default: `-1`
- **use_iterative_sort** – bool; Whether or not to use *squash_symmetry_redundant_sectors_sort()* with *iterative_sort()* to find sector symmetries. Default: `True`
- **use_light_Pak** – bool; Whether or not to use *squash_symmetry_redundant_sectors_sort()* with *light_Pak_sort()* to find sector symmetries. Default: `True`
- **use_dreadnaut** – bool or string, optional; Whether or not to use *squash_symmetry_redundant_sectors_dreadnaut()* to find sector symmetries. If given a string, interpret that string as the command line executable *dreadnaut*. If `True`, try `$SECDEC_CONTRIB/bin/dreadnaut` and, if the environment variable `$SECDEC_CONTRIB` is not set, `dreadnaut`. Default: `False`
- **use_Pak** – bool; Whether or not to use *squash_symmetry_redundant_sectors_sort()* with *Pak_sort()* to find sector symmetries. Default: `True`
- **processes** – integer or None, optional; The maximal number of processes to be used. If `None`, the number of CPUs `multiprocessing.cpu_count()` is used. *New in version 1.3*. Default: `None`

5.2.3 Drawing Feynman Diagrams

Use the following function to draw Feynman diagrams.

`pySecDec.loop_integral.draw.`**`plot_diagram`**`(`*internal_lines*, *external_lines*, *filename*, *powerlist=None*, *neato='neato'*, *extension='pdf'*, *Gstart=0*`)`

Draw a Feynman diagram using Graphviz (neato).

Thanks to Viktor Papara <papara@mpp.mpg.de> for his major contributions to this function.

Note: This function requires the command line tool *neato*. See also *Drawing Feynman Diagrams with neato*.

Warning: The target is overwritten without prompt if it exists already.

Parameters

- **internal_lines** – list; Adjacency list of internal lines, e.g. `[['m',['a',4]], ['m',[4,5]], ['m',['a',5]],[0,[1,2]],[0,[4,1]],[0,[2,5]]]`
- **external_lines** – list; Adjacency list of external lines, e.g. [['p1',1],['p2',2],['p3','a']]
- **filename** – string; The name of the output file. The generated file gets this name plus the file *extension*.
- **powerlist** – list, optional; The powers of the propagators defined by the *internal_lines*.
- **neato** – string, default: "neato"; The shell command to call "neato".
- **extension** – string, default: "pdf"; The file extension. This also defines the output format.
- **Gstart** – nonnegative int; The is value is passed to "neato" with the "-Gstart" option. Try changing this value if the visualization looks bad.

5.3 Polytope

The polytope class as required by `pySecDec.decomposition.geometric`.

class pySecDec.polytope.**Polytope**(*vertices=None*, *facets=None*)

Representation of a polytope defined by either its vertices or its facets. Call `complete_representation()` to translate from one to the other representation.

Parameters

- **vertices** – two dimensional array; The polytope in vertex representation. Each row is interpreted as one vertex.
- **facets** – two dimensional array; The polytope in facet representation. Each row represents one facet F. A row in *facets* is interpreted as one normal vector n_F with additionally the constant a_F in the last column. The points v of the polytope obey

$$\bigcap_F (\langle n_F, v\rangle + a_F) \geq 0$$

complete_representation(*normaliz='normaliz'*, *workdir='normaliz_tmp'*, *keep_workdir=False*)

Transform the vertex representation of a polytope to the facet representation or the other way round. Remove surplus entries in `self.facets` or `self.vertices`.

Note: This function calls the command line executable of *normaliz* [BIR]. See *The Geomethod and Normaliz* for installation and a list of tested versions.

Parameters

- **normaliz** – string; The shell command to run *normaliz*.
- **workdir** – string; The directory for the communication with *normaliz*. A directory with the specified name will be created in the current working directory. If the specified directory name already exists, an `OSError` is raised.

  Note: The communication with *normaliz* is done via files.

- **keep_workdir** – bool; Whether or not to delete the *workdir* after execution.

vertex_incidence_lists()

Return for each vertex the list of facets it lies in (as dictonary). The keys of the output dictonary are the vertices while the values are the indices of the facets in `self.facets`.

pySecDec.polytope.**convex_hull**(**polynomials*)

Calculate the convex hull of the Minkowski sum of all polynomials in the input. The algorithm sets all coefficients to one first and then only keeps terms of the polynomial product that have coefficient 1. Return the list of these entries in the expolist of the product of all input polynomials.

Parameters **polynomials** – abritrarily many instances of `Polynomial` where all of these have an equal number of variables; The polynomials to calculate the convex hull for.

pySecDec.polytope.**triangulate**(*cone*, *normaliz='normaliz'*, *workdir='normaliz_tmp'*, *keep_workdir=False*, *switch_representation=False*)

Split a cone into simplicial cones; i.e. cones defined by exactly D rays where D is the dimensionality.

Note: This function calls the command line executable of *normaliz* [BIR]. See *The Geomethod and Normaliz* for installation and a list of tested versions.

Parameters

- **cone** – two dimensional array; The defining rays of the cone.
- **normaliz** – string; The shell command to run *normaliz*.
- **workdir** – string; The directory for the communication with *normaliz*. A directory with the specified name will be created in the current working directory. If the specified directory name already exists, an `OSError` is raised.

  Note: The communication with *normaliz* is done via files.

- **keep_workdir** – bool; Whether or not to delete the *workdir* after execution.
- **switch_representation** – bool; Whether or not to switch between facet and vertex/ray representation.

5.4 Decomposition

The core of sector decomposition. This module implements the actual decomposition routines.

5.4.1 Common

This module collects routines that are used by multiple decompition modules.

class pySecDec.decomposition.**Sector**(*cast*, *other=[]*, *Jacobian=None*)
Container class for sectors that arise during the sector decomposition.

Parameters

- **cast** – iterable of *algebra.Product* or of *algebra.Polynomial*; The polynomials to be cast to the form *<monomial> * (const + . . .)*
- **other** – iterable of *algebra.Polynomial*, optional; All variable transformations are applied to these polynomials but it is not attempted to achieve the form *<monomial> * (const + . . .)*
- **Jacobian** – *algebra.Polynomial* with one term, optional; The Jacobian determinant of this sector. If not provided, the according unit monomial (1*x0^0*x1^0. . .) is assumed.

pySecDec.decomposition.**squash_symmetry_redundant_sectors_sort**(*sectors*, *sort_function*, *indices=None*)
Reduce a list of sectors by squashing duplicates with equal integral.

If two sectors only differ by a permutation of the polysymbols (to be interpreted as integration variables over some inteval), then the two sectors integrate to the same value. Thus we can drop one of them and count the other twice. The multiple counting of a sector is accounted for by increasing the coefficient of the Jacobian by one.

Equivalence up to permutation is established by applying the *sort_function* to each sector, this brings them into a canonical form. Sectors with identical canonical forms differ only by a permutation.

Note: whether all symmetries are found depends on the choice of *sort_function*. The sort function *pySecDec.matrix_sort.Pak_sort()* should find all symmetries whilst the sort functions *pySecDec.matrix_sort.iterative_sort()* and *pySecDec.matrix_sort.light_Pak_sort()* are faster but do not identify all symmetries.

See also: *squash_symmetry_redundant_sectors_dreadnaut()*

Example:

```
>>> from pySecDec.algebra import Polynomial
>>> from pySecDec.decomposition import Sector
>>> from pySecDec.decomposition import squash_symmetry_redundant_sectors_sort
>>> from pySecDec.matrix_sort import Pak_sort
>>>
>>> poly = Polynomial([(0,1),(1,0)], ['a','b'])
>>> swap = Polynomial([(1,0),(0,1)], ['a','b'])
>>> Jacobian_poly = Polynomial([(1,0)], [3]) # three
>>> Jacobian_swap = Polynomial([(0,1)], [5]) # five
>>> sectors = (
...               Sector([poly],Jacobian=Jacobian_poly),
...               Sector([swap],Jacobian=Jacobian_swap)
...           )
>>>
>>> reduced_sectors = squash_symmetry_redundant_sectors_sort(sectors,
...                   Pak_sort)
>>> len(reduced_sectors) # symmetry x0 <--> x1
1
>>> # The Jacobians are added together to account
```

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```
>>> # for the double counting of the sector.
>>> reduced_sectors[0].Jacobian
 + (8)*x0
```

Parameters

- **sectors** – iterable of *Sector*; the sectors to be reduced.
- **sort_function** – *pySecDec.matrix_sort.iterative_sort()*, *pySecDec.matrix_sort.light_Pak_sort()*, or *pySecDec.matrix_sort.Pak_sort()*; The function to be used for finding a canonical form of the sectors.
- **indices** – iterable of integers, optional; The indices of the variables to consider. If not provided, all indices are taken into account.

pySecDec.decomposition.**squash_symmetry_redundant_sectors_dreadnaut**(*sectors*, *indices=None*, *dreadnaut='dreadnaut'*, *workdir='dreadnaut_tmp'*, *keep_workdir=False*)

Reduce a list of sectors by squashing duplicates with equal integral.

Each *Sector* is converted to a *Polynomial* which is represented as a graph following the example of [MP+14] (v2.6 Figure 7, Isotopy of matrices).

We first multiply each polynomial in the sector by a unique tag then sum the polynomials of the sector, this converts a sector to a polynomial. Next, we convert the *expolist* of the resulting polynomial to a graph where each unique exponent in the *expolist* is considered to be a different symbol. Each unique coefficient in the polynomial's *coeffs* is assigned a vertex and connected to the row vertex of any term it multiplies. The external program *dreadnaut* is then used to bring the graph into a canonical form and provide a hash. Sectors with equivalent hashes may be identical, their canonical graphs are compared and if they are identical the sectors are combined.

Note: This function calls the command line executable of *dreadnaut* [MP+14]. It has been tested with *dreadnaut* version nauty26r7.

See also: *squash_symmetry_redundant_sectors_sort()*

Parameters

- **sectors** – iterable of *Sector*; the sectors to be reduced.
- **indices** – iterable of integers, optional; The indices of the variables to consider. If not provided, all indices are taken into account.
- **dreadnaut** – string; The shell command to run *dreadnaut*.
- **workdir** – string; The directory for the communication with *dreadnaut*. A directory with the specified name will be created in the current working directory. If the specified directory name already exists, an `OSError` is raised.

  Note: The communication with *dreadnaut* is done via files.

- **keep_workdir** – bool; Whether or not to delete the *workdir* after execution.

5.4.2 Iterative

The iterative sector decomposition routines.

exception pySecDec.decomposition.iterative.**EndOfDecomposition**

This exception is raised if the function *iteration_step()* is called although the sector is already in standard form.

pySecDec.decomposition.iterative.**find_singular_set**(*sector*, *indices=None*)

Function within the iterative sector decomposition procedure which heuristically chooses an optimal decomposition set. The strategy was introduced in arXiv:hep-ph/0004013 [BH00] and is described in 4.2.2 of arXiv:1410.7939 [Bor14]. Return a list of indices.

Parameters

- **sector** – *Sector*; The sector to be decomposed.
- **indices** – iterable of integers or None; The indices of the parameters to be considered as integration variables. By default (`indices=None`), all parameters are considered as integration variables.

pySecDec.decomposition.iterative.**iteration_step**(*sector*, *indices=None*)

Run a single step of the iterative sector decomposition as described in chapter 3.2 (part II) of arXiv:0803.4177v2 [Hei08]. Return an iterator of *Sector* - the arising subsectors.

Parameters

- **sector** – *Sector*; The sector to be decomposed.
- **indices** – iterable of integers or None; The indices of the parameters to be considered as integration variables. By default (`indices=None`), all parameters are considered as integration variables.

pySecDec.decomposition.iterative.**iterative_decomposition**(*sector*, *indices=None*)

Run the iterative sector decomposition as described in chapter 3.2 (part II) of arXiv:0803.4177v2 [Hei08]. Return an iterator of *Sector* - the arising subsectors.

Parameters

- **sector** – *Sector*; The sector to be decomposed.
- **indices** – iterable of integers or None; The indices of the parameters to be considered as integration variables. By default (`indices=None`), all parameters are considered as integration variables.

pySecDec.decomposition.iterative.**primary_decomposition**(*sector*, *indices=None*)

Perform the primary decomposition as described in chapter 3.2 (part I) of arXiv:0803.4177v2 [Hei08]. Return a list of *Sector* - the primary sectors. For *N* Feynman parameters, there are *N* primary sectors where the *i*-th Feynman parameter is set to *1* in sector *i*.

See also:

primary_decomposition_polynomial()

Parameters

- **sector** – *Sector*; The container holding the polynomials (typically U and F) to eliminate the Dirac delta from.
- **indices** – iterable of integers or None; The indices of the parameters to be considered as integration variables. By default (`indices=None`), all parameters are considered as integration variables.

`pySecDec.decomposition.iterative.`**`primary_decomposition_polynomial`**(*polynomial*, *indices=None*)

Perform the primary decomposition on a single polynomial.

See also:

primary_decomposition()

Parameters

- **polynomial** – *algebra.Polynomial*; The polynomial to eliminate the Dirac delta from.
- **indices** – iterable of integers or None; The indices of the parameters to be considered as integration variables. By default (`indices=None`), all parameters are considered as integration variables.

`pySecDec.decomposition.iterative.`**`remap_parameters`**(*singular_parameters*, *Jacobian*, **polynomials*)

Remap the Feynman parameters according to eq. (16) of arXiv:0803.4177v2 [Hei08]. The parameter whose index comes first in *singular_parameters* is kept fix.

The remapping is done in place; i.e. the *polynomials* are **NOT** copied.

Parameters

- **singular_parameters** – list of integers; The indices α_r such that at least one of *polynomials* becomes zero if all $t_{\alpha_r} \to 0$.
- **Jacobian** – *Polynomial*; The Jacobian determinant is multiplied to this polynomial.
- **polynomials** – abritrarily many instances of *algebra.Polynomial* where all of these have an equal number of variables; The polynomials of Feynman parameters to be remapped. These are typically F and U.

Example:

```
remap_parameters([1,2], Jacobian, F, U)
```

5.4.3 Geometric

The geometric sector decomposition routines.

`pySecDec.decomposition.geometric.`**`Cheng_Wu`**(*sector*, *index=-1*)

Replace one Feynman parameter by one. This means integrating out the Dirac delta according to the Cheng-Wu theorem.

Parameters

- **sector** – *Sector*; The container holding the polynomials (typically U and F) to eliminate the Dirac delta from.
- **index** – integer, optional; The index of the Feynman parameter to eliminate. Default: -1 (the last Feynman parameter)

`pySecDec.decomposition.geometric.`**`generate_fan`**(**polynomials*)

Calculate the fan of the polynomials in the input. The rays of a cone are given by the exponent vectors after factoring out a monomial together with the standard basis vectors. Each choice of factored out monomials gives a different cone. Only full (N-) dimensional cones in $R^N_{\geq 0}$ need to be considered.

Parameters **polynomials** – abritrarily many instances of *Polynomial* where all of these have an equal number of variables; The polynomials to calculate the fan for.

`pySecDec.decomposition.geometric.`**`geometric_decomposition`**(*sector*, *indices=None*, *normaliz='normaliz'*, *workdir='normaliz_tmp'*)

Run the sector decomposition using the geomethod as described in [BHJ+15].

Note: This function calls the command line executable of *normaliz* [BIR]. See *The Geomethod and Normaliz* for installation and a list of tested versions.

Parameters

- **sector** – *Sector*; The sector to be decomposed.
- **indices** – list of integers or None; The indices of the parameters to be considered as integration variables. By default (`indices=None`), all parameters are considered as integration variables.
- **normaliz** – string; The shell command to run *normaliz*.
- **workdir** – string; The directory for the communication with *normaliz*. A directory with the specified name will be created in the current working directory. If the specified directory name already exists, an `OSError` is raised.

Note: The communication with *normaliz* is done via files.

`pySecDec.decomposition.geometric.`**`geometric_decomposition_ku`**(*sector*, *indices=None*, *normaliz='normaliz'*, *workdir='normaliz_tmp'*)

Run the sector decomposition using the original geometric decomposition strategy by Kaneko and Ueda as described in [KU10].

Note: This function calls the command line executable of *normaliz* [BIR]. See *The Geomethod and Normaliz* for installation and a list of tested versions.

Parameters

- **sector** – *Sector*; The sector to be decomposed.
- **indices** – list of integers or None; The indices of the parameters to be considered as integration variables. By default (`indices=None`), all parameters are considered as integration variables.
- **normaliz** – string; The shell command to run *normaliz*.
- **workdir** – string; The directory for the communication with *normaliz*. A directory with the specified name will be created in the current working directory. If the specified directory name already exists, an `OSError` is raised.

Note: The communication with *normaliz* is done via files.

`pySecDec.decomposition.geometric.`**`transform_variables`**(*polynomial*, *transformation*, *polysymbols='y'*)

Transform the parameters x_i of a *`pySecDec.algebra.Polynomial`*,

$$x_i \to \prod_j x_j^{T_{ij}}$$

, where T_{ij} is the transformation matrix.

Parameters

- **`polynomial`** – *`pySecDec.algebra.Polynomial`*; The polynomial to transform the variables in.
- **`transformation`** – two dimensional array; The transformation matrix T_{ij}.
- **`polysymbols`** – string or iterable of strings; The symbols for the new variables. This argument is passed to the default constructor of *`pySecDec.algebra.Polynomial`*. Refer to the documentation of *`pySecDec.algebra.Polynomial`* for further details.

5.4.4 Splitting

Routines to split the integration between 0 and 1. This maps singularities from 1 to 0.

`pySecDec.decomposition.splitting.`**`find_singular_sets_at_one`**(*polynomial*)

Find all possible sets of parameters such that the *polynomial*'s constant term vanishes if these parameters are set to one.

Example:

```
>>> from pySecDec.algebra import Polynomial
>>> from pySecDec.decomposition.splitting import find_singular_sets_at_one
>>> polysymbols = ['x0', 'x1']
>>> poly = Polynomial.from_expression('1 - 10*x0 - x1', polysymbols)
>>> find_singular_sets_at_one(poly)
[(1,)]
```

Parameters **`polynomial`** – *`Polynomial`*; The polynomial to search in.

`pySecDec.decomposition.splitting.`**`remap_one_to_zero`**(*polynomial*, **indices*)

Apply the transformation $x \to 1 - x$ to *polynomial* for the parameters of the given *indices*.

Parameters

- **`polynomial`** – *`Polynomial`*; The polynomial to apply the transformation to.
- **`indices`** – arbitrarily many int; The indices of the `polynomial.polysymbols` to apply the transformation to.

Example:

```
>>> from pySecDec.algebra import Polynomial
>>> from pySecDec.decomposition.splitting import remap_one_to_zero
>>> polysymbols = ['x0']
>>> polynomial = Polynomial.from_expression('x0', polysymbols)
>>> remap_one_to_zero(polynomial, 0)
 + (1) + (-1)*x0
```

pySecDec.decomposition.splitting.**split**(*sector*, *seed*, **indices*)

Split the integration interval $[0, 1]$ for the parameters given by *indices*. The splitting point is fixed using *numpy's* random number generator.

Return an iterator of *Sector* - the arising subsectors.

Parameters **sector** – *Sector*; The sector to be split.

:param seed; integer; The seed for the random number generator that is used to fix the splitting point.

Parameters **indices** – arbitrarily many integers; The indices of the variables to be split.

pySecDec.decomposition.splitting.**split_singular**(*sector*, *seed*, *indices=[]*)

Split the integration interval $[0, 1]$ for the parameters that can lead to singularities at one for the polynomials in `sector.cast`.

Return an iterator of *Sector* - the arising subsectors.

Parameters

- **sector** – *Sector*; The sector to be split.
- **seed** – integer; The seed for the random number generator that is used to fix the splitting point.
- **indices** – iterables of integers; The indices of the variables to be split if required. An empty iterator means that all variables may potentially be split.

5.5 Matrix Sort

Algorithms to sort a matrix when column and row permutations are allowed.

pySecDec.matrix_sort.**Pak_sort**(*matrix*, **indices*)

Inplace modify the *matrix* to some canonical ordering, when permutations of rows and columns are allowed.

The *indices* parameter can contain a list of lists of column indices. Only the columns present in the same list are swapped with each other.

The implementation of this function is described in chapter 2 of [Pak11].

Note: If not all indices are considered the resulting matrix may not be canonical.

See also:

iterative_sort(), *light_Pak_sort()*

Parameters

- **matrix** – 2D array-like; The matrix to be canonicalized.
- **indices** – arbitrarily many iterables of non-negative integers; The groups of columns to permute. Default: `range(1,matrix.shape[1])`

pySecDec.matrix_sort.**iterative_sort**(*matrix*)

Inplace modify the *matrix* to some ordering, when permutations of rows and columns (excluding the first) are allowed.

Note: This function may result in different orderings depending on the initial ordering.

See also:

Pak_sort(), *light_Pak_sort()*

Parameters **matrix** – 2D array-like; The matrix to be canonicalized.

pySecDec.matrix_sort.**light_Pak_sort**(*matrix*)

Inplace modify the *matrix* to some ordering, when permutations of rows and columns (excluding the first) are allowed. The implementation of this function is described in chapter 2 of [Pak11]. This function implements a lightweight version: In step (v), we only consider one, not all table copies with the minimized second column.

Note: This function may result in different orderings depending on the initial ordering.

See also:

iterative_sort(), *Pak_sort()*

Parameters **matrix** – 2D array-like; The matrix to be canonicalized.

5.6 Subtraction

Routines to isolate the divergencies in an ϵ expansion.

pySecDec.subtraction.**integrate_by_parts**(*polyprod*, *power_goals*, *indices*)

Repeatedly apply integration by parts,

$$\int_0^1 dt_j t_j^{(a_j - b_j\epsilon_1 - c\epsilon_2 + \ldots)} \mathcal{I}(t_j, \{t_{i\neq j}\}, \epsilon_1, \epsilon_2, \ldots) = \frac{1}{a_j + 1 - b_j\epsilon_1 - c\epsilon_2 - \ldots} \left(\mathcal{I}(1, \{t_{i\neq j}\}, \epsilon_1, \epsilon_2, \ldots) - \int_0^1 dt_j t_j^{(a_j + 1 - b_j\epsilon_1 - c\epsilon_2 +$$

, where $\mathcal{I}'$ denotes the derivative of $\mathcal{I}$ with respect to t_j. The iteration stops, when $a_j >=$ *power_goal_j*.

See also:

This function provides an alternative to *integrate_pole_part()*.

Parameters

- **polyprod** – *algebra.Product* of the form <product of <monomial>**(a_j + ...)> * <regulator poles of cal_I> * <cal_I>; The input product as decribed above. The <product of <monomial>**(a_j + ...)> should be a *pySecDec.algebra.Product* of <monomial>**(a_j + ...). as described below. The <monomial>**(a_j + ...) should be an *pySecDec.algebra.ExponentiatedPolynomial* with exponent being a *Polynomial* of the regulators $\epsilon_1, \epsilon_2, \ldots$. Although no dependence on the Feynman parameters is expected in the exponent, the polynomial variables should be the Feynman parameters and the regulators. The constant term of the exponent should be numerical. The polynomial variables of monomial and the other factors (interpreted as $\mathcal{I}$) are interpreted as the Feynman parameters and the epsilon regulators. Make sure that the last factor (<cal_I>) is defined and finite for $\epsilon = 0$. All poles for $\epsilon \to 0$ should be made explicit by putting them into <regulator poles of cal_I> as *pySecDec.algebra.Pow* with exponent = -1 and the base of type *pySecDec.algebra.Polynomial*.

- **power_goals** – number or iterable of numbers, e.g. float, integer, ...; The stopping criterion for the iteration.
- **indices** – iterable of integers; The index/indices of the parameter(s) to partially integrate. j in the formulae above.

Return the pole part and the numerically integrable remainder as a list. Each returned list element has the same structure as the input *polyprod*.

`pySecDec.subtraction.`**`integrate_pole_part`**`(`*polyprod*, **indices*`)`

Transform an integral of the form

$$\int_0^1 dt_j t_j^{(a-b\epsilon_1-c\epsilon_2+...)} \mathcal{I}(t_j, \{t_{i\neq j}\}, \epsilon_1, \epsilon_2, ...)$$

into the form

$$\sum_{p=0}^{|a|-1} \frac{1}{a+p+1-b\epsilon_1-c\epsilon_2-...} \frac{\mathcal{I}^{(p)}(0, \{t_{i\neq j}\}, \epsilon_1, \epsilon_2, ...)}{p!} + \int_0^1 dt_j t_j^{(a-b\epsilon_1-c\epsilon_2+...)} R(t_j, \{t_{i\neq j}\}, \epsilon_1, \epsilon_2, ...)$$

, where $\mathcal{I}^{(p)}$ denotes the p-th derivative of $\mathcal{I}$ with respect to t_j. The equations above are to be understood schematically.

See also:

This function implements the transformation from equation (19) to (21) as described in arXiv:0803.4177v2 [Hei08].

Parameters

- **polyprod** – *algebra.Product* of the form <product of <monomial>**(a_j + ...)> * <regulator poles of cal_I> * <cal_I>; The input product as decribed above. The <product of <monomial>**(a_j + ...)> should be a *pySecDec.algebra.Product* of <monomial>**(a_j + ...). as described below. The <monomial>**(a_j + ...) should be an *pySecDec.algebra.ExponentiatedPolynomial* with `exponent` being a *Polynomial* of the regulators $\epsilon_1, \epsilon_2, ...$. Although no dependence on the Feynman parameters is expected in the `exponent`, the polynomial variables should be the Feynman parameters and the regulators. The constant term of the exponent should be numerical. The polynomial variables of `monomial` and the other factors (interpreted as $\mathcal{I}$) are interpreted as the Feynman parameters and the epsilon regulators. Make sure that the last factor (<cal_I>) is defined and finite for $\epsilon = 0$. All poles for $\epsilon \to 0$ should be made explicit by putting them into <regulator poles of cal_I> as *pySecDec.algebra.Pow* with `exponent = -1` and the `base` of type *pySecDec.algebra.Polynomial*.
- **indices** – arbitrarily many integers; The index/indices of the parameter(s) to partially integrate. j in the formulae above.

Return the pole part and the numerically integrable remainder as a list. That is the sum and the integrand of equation (21) in arXiv:0803.4177v2 [Hei08]. Each returned list element has the same structure as the input *polyprod*.

`pySecDec.subtraction.`**`pole_structure`**`(`*monomial_product*, **indices*`)`

Return a list of the unregulated exponents of the parameters specified by *indices* in *monomial_product*.

Parameters

- **monomial_product** – *pySecDec.algebra.ExponentiatedPolynomial* with `exponent` being a *Polynomial*; The monomials of the subtraction to extract the pole structure from.

- **indices** – arbitrarily many integers; The index/indices of the parameter(s) to partially investigate.

5.7 Expansion

Routines to series expand singular and nonsingular expressions.

exception `pySecDec.expansion.`**OrderError**

This exception is raised if an expansion to a lower than the lowest order of an expression is requested.

`pySecDec.expansion.`**expand_Taylor**(*expression, indices, orders*)

Series/Taylor expand a nonsingular *expression* around zero.

Return a *algebra.Polynomial* - the series expansion.

Parameters

- **expression** – an expression composed of the types defined in the module *algebra*; The expression to be series expanded.
- **indices** – integer or iterable of integers; The indices of the parameters to expand. The ordering of the indices defines the ordering of the expansion.
- **order** – integer or iterable of integers; The order to which the expansion is to be calculated.

`pySecDec.expansion.`**expand_singular**(*product, indices, orders*)

Series expand a potentially singular expression of the form

$$\frac{a_N \epsilon_0 + b_N \epsilon_1 + ...}{a_D \epsilon_0 + b_D \epsilon_1 + ...}$$

Return a *algebra.Polynomial* - the series expansion.

See also:

To expand more general expressions use *expand_sympy()*.

Parameters

- **product** – *algebra.Product* with factors of the form `<polynomial>` and `<polynomial> ** -1`; The expression to be series expanded.
- **indices** – integer or iterable of integers; The indices of the parameters to expand. The ordering of the indices defines the ordering of the expansion.
- **order** – integer or iterable of integers; The order to which the expansion is to be calculated.

`pySecDec.expansion.`**expand_sympy**(*expression, variables, orders*)

Expand a sympy expression in the *variables* to given *orders*. Return the expansion as nested *pySecDec.algebra.Polynomial*.

See also:

This function is a generalization of *expand_singular()*.

Parameters

- **expression** – string or sympy expression; The expression to be expanded
- **variables** – iterable of strings or sympy symbols; The variables to expand the *expression* in.

- **orders** – iterable of integers; The orders to expand to.

5.8 Code Writer

This module collects routines to create a c++ library.

5.8.1 Make Package

This is the main function of *pySecDec*.

`pySecDec.code_writer.`**`make_package`**(*name*, *integration_variables*, *regulators*, *requested_orders*, *polynomials_to_decompose*, *polynomial_names=[]*, *other_polynomials=[]*, *prefactor=1*, *remainder_expression=1*, *functions=[]*, *real_parameters=[]*, *complex_parameters=[]*, *form_optimization_level=2*, *form_work_space='500M'*, *form_insertion_depth=5*, *contour_deformation_polynomial=None*, *positive_polynomials=[]*, *decomposition_method='iterative_no_primary'*, *normaliz_executable='normaliz'*, *enforce_complex=False*, *split=False*, *ibp_power_goal=-1*, *use_iterative_sort=True*, *use_light_Pak=True*, *use_dreadnaut=False*, *use_Pak=True*, *processes=None*)

Decompose, subtract and expand an expression. Return it as c++ package.

See also:

In order to decompose a loop integral, use the function `pySecDec.loop_integral.loop_package()`.

See also:

The generated library is described in *Generated C++ Libraries*.

Parameters

- **name** – string; The name of the c++ namepace and the output directory.
- **integration_variables** – iterable of strings or sympy symbols; The variables that are to be integrated from 0 to 1.
- **regulators** – iterable of strings or sympy symbols; The UV/IR regulators of the integral.
- **requested_orders** – iterable of integers; Compute the expansion in the regulators to these orders.
- **polynomials_to_decompose** – iterable of strings or sympy expressions or `pySecDec.algebra.ExponentiatedPolynomial` or `pySecDec.algebra.Polynomial`; The polynomials to be decomposed.
- **polynomial_names** – iterable of strings; Assign symbols for the *polynomials_to_decompose*. These can be referenced in the *other_polynomials*; see *other_polynomials* for details.
- **other_polynomials** – iterable of strings or sympy expressions or `pySecDec.algebra.ExponentiatedPolynomial` or `pySecDec.algebra.Polynomial`; Additional polynomials where no decomposition is attempted. The symbols defined in *polynomial_names* can be used to reference the *polynomials_to_decompose*. This is particularly

useful when computing loop integrals where the "numerator" can depend on the first and second Symanzik polynomials.

Example (1-loop bubble with numerator):

```
>>> polynomials_to_decompose = ["(x0 + x1)**(2*eps - 4)",
...                             "(-p**2*x0*x1)**(-eps))"]
>>> polynomial_names = ["U", "F"]
>>> other_polynomials = ["""   (eps - 1)*s*U**2
...                         + (eps - 2)*F
...                         - (eps - 1)*2*s*x0*U
...                         + (eps - 1)*s*x0**2"""]
```

See also:

pySecDec.loop_integral

Note that the *polynomial_names* refer to the *polynomials_to_decompose* **without** their exponents.

- **prefactor** – string or sympy expression, optional; A factor that does not depend on the integration variables.
- **remainder_expression** – string or sympy expression or `pySecDec.algebra._Expression`, optional; An additional factor.

  Dummy function must be provided with all arguments, e.g. `remainder_expression='exp(eps)*f(x0,x1)'`. In addition, all dummy function must be listed in *functions*.
- **functions** – iterable of strings or sympy symbols, optional; Function symbols occuring in *remainder_expression*, e.g.``['f']``.

Note: Only user-defined functions that are provided as c++-callable code should be mentioned here. Listing basic mathematical functions (e.g. `log`, `pow`, `exp`, `sqrt`, . . .) is not required and considered an error to avoid name conflicts.

Note: The power function *pow* and the logarithm *log* use the nonstandard continuation with an infinitesimal negative imaginary part on the negative real axis (e.g. `log(-1) = -i*pi`).

- **real_parameters** – iterable of strings or sympy symbols, optional; Symbols to be interpreted as real variables.
- **complex_parameters** – iterable of strings or sympy symbols, optional; Symbols to be interpreted as complex variables.
- **form_optimization_level** – integer out of the interval [0,4], optional; The optimization level to be used in FORM. Default: `2`.
- **form_work_space** – string, optional; The FORM WorkSpace. Default: `'500M'`.
- **form_insertion_depth** – nonnegative integer, optional; How deep FORM should try to resolve nested function calls. Default: `5`.
- **contour_deformation_polynomial** – string or sympy symbol, optional; The name of the polynomial in *polynomial_names* that is to be continued to the complex plane accord-

ing to a $-i\delta$ prescription. For loop integrals, this is the second Symanzik polynomial `F`. If not provided, no code for contour deformation is created.

- **positive_polynomials** – iterable of strings or sympy symbols, optional; The names of the polynomials in *polynomial_names* that should always have a positive real part. For loop integrals, this applies to the first Symanzik polynomial `U`. If not provided, no polynomial is checked for positiveness. If *contour_deformation_polynomial* is `None`, this parameter is ignored.
- **decomposition_method** – string, optional; The strategy to decompose the polynomials. The following strategies are available:
  - 'iterative_no_primary' (default)
  - 'geometric_no_primary'
  - 'iterative'
  - 'geometric'
  - 'geometric_ku'

  'iterative', 'geometric', and 'geometric_ku' are only valid for loop integrals. An end user should always use 'iterative_no_primary' or 'geometric_no_primary' here. In order to compute loop integrals, please use the function *pySecDec.loop_integral.loop_package()*.
- **normaliz_executable** – string, optional; The command to run *normaliz*. *normaliz* is only required if *decomposition_method* starts with 'geometric'. Default: 'normaliz'
- **enforce_complex** – bool, optional; Whether or not the generated integrand functions should have a complex return type even though they might be purely real. The return type of the integrands is automatically complex if *contour_deformation* is `True` or if there are *complex_parameters*. In other cases, the calculation can typically be kept purely real. Most commonly, this flag is needed if `log(<negative real>)` occurs in one of the integrand functions. However, *pySecDec* will suggest setting this flag to `True` in that case. Default: `False`
- **split** – bool or integer, optional; Whether or not to split the integration domain in order to map singularities from 1 to 0. Set this option to `True` if you have singularties when one or more integration variables are one. If an integer is passed, that integer is used as seed to generate the splitting point. Default: `False`
- **ibp_power_goal** – number or iterable of number, optional; The *power_goal* that is forwarded to *integrate_by_parts()*.

  This option controls how the subtraction terms are generated. Setting it to `-numpy.inf` disables *integrate_by_parts()*, while `0` disables *integrate_pole_part()*.

  See also:

  To generate the subtraction terms, this function first calls *integrate_by_parts()* for each integration variable with the give *ibp_power_goal*. Then *integrate_pole_part()* is called.

  Default: `-1`
- **use_iterative_sort** – bool; Whether or not to use *squash_symmetry_redundant_sectors_sort()* with *iterative_sort()* to find sector symmetries. Default: `True`

- **use_light_Pak** – bool; Whether or not to use *squash_symmetry_redundant_sectors_sort()* with *light_Pak_sort()* to find sector symmetries. Default: `True`
- **use_dreadnaut** – bool or string, optional; Whether or not to use *squash_symmetry_redundant_sectors_dreadnaut()* to find sector symmetries. If given a string, interpret that string as the command line executable *dreadnaut*. If `True`, try `$SECDEC_CONTRIB/bin/dreadnaut` and, if the environment variable `$SECDEC_CONTRIB` is not set, `dreadnaut`. Default: `False`
- **use_Pak** – bool; Whether or not to use *squash_symmetry_redundant_sectors_sort()* with *Pak_sort()* to find sector symmetries. Default: `True`
- **processes** – integer or None, optional; The maximal number of processes to be used. If `None`, the number of CPUs `multiprocessing.cpu_count()` is used. *New in version 1.3*. Default: `None`

5.8.2 Template Parser

Functions to generate c++ sources from template files.

`pySecDec.code_writer.template_parser.`**`parse_template_file`**`(`*src*, *dest*, *replacements={}*`)`

Copy a file from *src* to *dest* replacing `%(...)` instructions in the standard python way.

Warning: If the file specified in *dest* exists, it is overwritten without prompt.

See also:

parse_template_tree()

Parameters

- **src** – str; The path to the template file.
- **dest** – str; The path to the destination file.
- **replacements** – dict; The replacements to be performed. The standard python replacement rules apply:

```
>>> '%(var)s = %(value)i' % dict(
...     var = 'my_variable',
...     value = 5)
'my_variable = 5'
```

`pySecDec.code_writer.template_parser.`**`parse_template_tree`**`(`*src*, *dest*, *replacements_in_files={}*, *filesystem_replacements={}*`)`

Copy a directory tree from *src* to *dest* using *parse_template_file()* for each file and replacing the filenames according to *filesystem_replacements*.

See also:

parse_template_file()

Parameters

- **src** – str; The path to the template directory.
- **dest** – str; The path to the destination directory.
- **replacements_in_files** – dict; The replacements to be performed in the files. The standard python replacement rules apply:

```
>>> '%(var)s = %(value)i' % dict(
...     var = 'my_variable',
...     value = 5)
'my_variable = 5'
```

- **filesystem_replacements** – dict; Renaming rules for the destination files. and directories. If a file or directory name in the source tree *src* matches a key in this dictionary, it is renamed to the corresponding value. If the value is `None`, the corresponding file is ignored.

5.9 Generated C++ Libraries

A C++ Library to numerically compute a given integral (loop integral) can be generated by the `make_package()` (`loop_package()`) functions. The *name* passed to the `make_package()` or `loop_package()` function will be used as the C++ namespace of the generated library. A program demonstrating the use of the C++ library is generated for each integral and written to `name/integrate_name.cpp`. Here we document the C++ library API.

See also:

C++ Interface

typedef double **real_t**
The real type used by the library.

typedef std::complex<*real_t*> **complex_t**
The complex type used by the library.

type integrand_return_t
The return type of the integrand function. If the integral has complex parameters or uses contour deformation or if *enforce_complex* is set to `True` in the call to `make_package()` or `loop_package()` then *integrand_return_t* is *complex_t*. Otherwise *integrand_return_t* is *real_t*.

template<typename **T**>
using nested_series_t = secdecutil::*Series*<secdecutil::*Series*<...<T>>>
A potentially nested `secdecutil::Series` representing the series expansion in each of the regulators. If the integral depends on only one regulator (for example, a loop integral generated with `loop_package()`) this type will be a `secdecutil::Series`. For integrals that depend on multiple regulators then this will be a series of series representing the multivariate series. This type can be used to write code that can handle integrals depending on arbitrarily many regulators.

See also:

`secdecutil::Series`

typedef secdecutil::*IntegrandContainer*<*integrand_return_t*, *real_t* **const** ***const** > **integrand_t**
The type of the integrand. Within the generated C++ library integrands are stored in a container along with the number of integration variables upon which they depend. These containers can be passed to an integrator for numerical integration.

See also:

`secdecutil::IntegrandContainer` and `secdecutil::Integrator`.

type cuda_integrand_t
New in version 1.4.

The type of a single integrand (sector) usable on a CUDA device (GPU). This container can be passed to an integrator for numerical integration.

See also:

secdecutil::IntegrandContainer, *secdecutil::Integrator*, and *secdecutil::integrators::Qmc*.

type cuda_together_integrand_t
New in version 1.4.

The type of a sum of integrands (sectors) usable on a CUDA device (GPU). This container can be passed to an integrator for numerical integration.

See also:

secdecutil::IntegrandContainer, *secdecutil::Integrator*, and *secdecutil::integrators::Qmc*.

const unsigned long long **number_of_sectors**
The number of sectors generated by the sector decomposition.

Changed in version 1.3.1: Type was `unsigned int` in earlier versions of *pySecDec*.

const unsigned int **maximal_number_of_integration_variables**
The number of integration variables after primary decomposition. This provides an upper bound in the number of integration variables for all integrand functions. The actual number of integration variables may be lower for a given integrand.

const unsigned int **number_of_regulators**
The number of regulators on which the integral depends.

const unsigned int **number_of_real_parameters**
The number of real parameters on which the integral depends.

const std::vector<std::string> **names_of_real_parameters**
An ordered vector of string representations of the names of the real parameters.

const unsigned int **number_of_complex_parameters**
The number of complex parameters on which the integral depends.

const std::vector<std::string> **names_of_complex_parameters**
An ordered vector of string representations of the names of the complex parameters.

const std::vector<int> **lowest_orders**
A vector of the lowest order of each regulator which appears in the integral, not including the prefactor.

const std::vector<int> **highest_orders**
A vector of the highest order of each regulator which appears in the integral, not including the prefactor. This depends on the *requested_orders* and *prefactor/additional_prefactor* parameter passed to `make_package()` or `loop_package()`. In the case of `loop_package()` it also depends on the Γ-function prefactor of the integral which appears upon Feynman parametrization.

const std::vector<int> **lowest_prefactor_orders**
A vector of the lowest order of each regulator which appears in the prefactor of the integral.

const std::vector<int> **highest_prefactor_orders**
A vector of the highest order of each regulator which appears in the prefactor of the integral.

const std::vector<int> **requested_orders**

A vector of the requested orders of each regulator used to generate the C++ library, i.e. the *requested_orders* parameter passed to `make_package()` or `loop_package()`.

const std::vector<*nested_series_t*<sector_container_t>> &**get_sectors**()

Changed in version 1.3.1: The variable `sectors` has been replaced by this function.

A low level interface for obtaining the underlying integrand C++ functions.

> **Warning:** The precise definition and usage of `get_sectors()` is likely to change in future versions of *pySecDec*.

nested_series_t<*integrand_return_t*> **prefactor**(**const** std::vector<*real_t*> &*real_parameters*, **const** std::vector<*complex_t*> &*complex_parameters*)

The series expansion of the integral prefactor evaluated with the given parameters. If the library was generated using `make_package()` it will be equal to the *prefactor* passed to `make_package()`. If the library was generated with `loop_package()` it will be the product of the *additional_prefactor* passed to `loop_package()` and the Γ-function prefactor of the integral which appears upon Feynman parametrization.

const std::vector<std::vector<*real_t*>> **pole_structures**

A vector of the powers of the monomials that can be factored out of each sector of the polynomial during the decomposition.

Example: an integral depending on variables x and y may have two sectors, the first may have a monomial $x^{-1}y^{-2}$ factored out and the second may have a monomial x^{-1} factored out during the decomposition. The resulting *pole_structures* would read `{ {-1,-2}, {-1,0} }`. Poles of type x^{-1} are known as logarithmic poles, poles of type x^{-2} are known as linear poles.

std::vector<*nested_series_t*<*integrand_t*>> **make_integrands**(**const** std::vector<*real_t*> &*real_parameters*, **const** std::vector<*complex_t*> &*complex_parameters*)

(without contour deformation)

std::vector<*nested_series_t*<*cuda_integrand_t*>> **make_cuda_integrands**(**const** std::vector<*real_t*> &*real_parameters*, **const** std::vector<*complex_t*> &*complex_parameters*)

New in version 1.4.

(without contour deformation) (CUDA only)

std::vector<*nested_series_t*<*integrand_t*>> **make_integrands**(**const** std::vector<*real_t*> &*real_parameters*, **const** std::vector<*complex_t*> &*complex_parameters*, unsigned *number_of_presamples* = 100000, *real_t* *deformation_parameters_maximum* = 1., *real_t* *deformation_parameters_minimum* = 1.e-5, *real_t* *deformation_parameters_decrease_factor* = 0.9)

(with contour deformation)

std::vector<*nested_series_t*<*cuda_integrand_t*>> **make_cuda_integrands**(**const** std::vector<*real_t*> &*real_parameters*, **const** std::vector<*complex_t*> &*complex_parameters*, unsigned *number_of_presamples* = 100000, *real_t* *deformation_parameters_maximum* = 1., *real_t* *deformation_parameters_minimum* = 1.e-5, *real_t* *deformation_parameters_decrease_factor* = 0.9)

New in version 1.4.

(with contour deformation) (CUDA only)

Gives a vector containing the series expansions of individual sectors of the integrand after sector decomposition with the specified *real_paraemters* and *complex_parameters* bound. Each element of the vector contains the series expansion of an individual sector. The series consists of instances of `integrand_t` (`cuda_integrand_t`) which contain the integrand functions and the number of integration variables upon which they depend. The real and complex parameters are bound to the values passed in *real_parameters* and *complex_parameters*. If enabled, contour deformation is controlled by the parameters *number_of_presamples*, *deformation_parameters_maximum*, *deformation_parameters_minimum*, *deformation_parameters_decrease_factor* which are documented in `pySecDec.integral_interface.IntegralLibrary`. In case of a sign check error (sign_check_error), manually set *number_of_presamples*, *deformation_parameters_maximum*, and *deformation_parameters_minimum*.

Passing the *integrand_t* to the `secdecutil::Integrator::integrate()` function of an instance of a particular `secdecutil::Integrator` will return the numerically evaluated integral. To integrate all orders of all sectors `secdecutil::deep_apply()` can be used.

Note: This is the recommended way to access the integrand functions.

See also:

C++ Interface, *Integrator Examples*, `pySecDec.integral_interface.IntegralLibrary`

5.10 Integral Interface

An interface to libraries generated by `pySecDec.code_writer.make_package()` or `pySecDec.loop_integral.loop_package()`.

class pySecDec.integral_interface.**CPPIntegrator**

Abstract base class for integrators to be used with an `IntegralLibrary`. This class holds a pointer to the c++ integrator and defines the destructor.

class pySecDec.integral_interface.**CQuad**(*integral_library*, *epsrel=0.01*, *epsabs=1e-07*, *n=100*, *verbose=False*, *zero_border=0.0*)

Wrapper for the cquad integrator defined in the gsl library.

Parameters integral_library – `IntegralLibrary`; The integral to be computed with this integrator.

The other options are defined in Section 4.5.1 and in the gsl manual.

class pySecDec.integral_interface.**CudaQmc**(*integral_library*, *transform*, *fitfunction='default'*, *generatingvectors='default'*, *epsrel=0.0*, *epsabs=0.0*, *maxeval=0*, *errormode='default'*, *evaluateminn=0*, *minn=0*, *minm=0*, *maxnperpackage=0*, *maxmperpackage=0*, *cputhreads=0*, *cudablocks=0*, *cudathreadsperblock=0*, *verbosity=0*, *seed=0*, *devices=[]*)

Wrapper for the Qmc integrator defined in the integrators library for GPU use.

Parameters

- **integral_library** – *IntegralLibrary*; The integral to be computed with this integrator.
- **errormode** – string; The *errormode* parameter of the Qmc, can be `"default"`, `"all"`, and `"largest"`. `"default"` takes the default from the Qmc library. See the Qmc docs for details on the other settings.
- **transform** – string; An integral transform related to the parameter *P* of the Qmc. The possible choices correspond to the integral transforms of the underlying Qmc implementation. Possible values are, `"none"`, `"baker"`, `sidi#`, `"korobov#"`, and `korobov#x#` where any `#` (the rank of the Korobov/Sidi transform) must be an integer between 1 and 6.
- **fitfunction** – string; An integral transform related to the parameter *F* of the Qmc. The possible choices correspond to the integral transforms of the underlying Qmc implementation. Possible values are `"default"`, `"none"`, `"polysingular"`.
- **generatingvectors** – string; The name of a set of generating vectors. The possible choices correspond to the available generating vectors of the underlying Qmc implementation. Possible values are `"default"`, `"cbcpt_dn1_100"`, `"cbcpt_dn2_6"` and `"cbcpt_cfftw1_6"`.

The other options are defined in the Qmc docs. If an argument is set to 0 then the default of the underlying Qmc implementation is used.

class pySecDec.integral_interface.**Cuhre**(*integral_library*, *epsrel=0.01*, *epsabs=1e-07*, *flags=0*, *mineval=0*, *maxeval=1000000*, *zero_border=0.0*, *key=0*, *real_complex_together=False*)

Wrapper for the Cuhre integrator defined in the cuba library.

Parameters integral_library – *IntegralLibrary*; The integral to be computed with this integrator.

The other options are defined in Section 4.5.3 and in the cuba manual.

class pySecDec.integral_interface.**Divonne**(*integral_library*, *epsrel=0.01*, *epsabs=1e-07*, *flags=0*, *seed=0*, *mineval=0*, *maxeval=1000000*, *zero_border=0.0*, *key1=2000*, *key2=1*, *key3=1*, *maxpass=4*, *border=0.0*, *maxchisq=1.0*, *mindeviation=0.15*, *real_complex_together=False*)

Wrapper for the Divonne integrator defined in the cuba library.

Parameters integral_library – *IntegralLibrary*; The integral to be computed with this integrator.

The other options are defined in Section 4.5.3 and in the cuba manual.

class pySecDec.integral_interface.**IntegralLibrary**(*shared_object_path*)

Interface to a c++ library produced by *make_package()* or *loop_package()*.

Parameters **shared_object_path** – str; The path to the file "<name>_pylink.so" that can be built by the command

```
$ make pylink
```

in the root directory of the c++ library.

Instances of this class can be called with the following arguments:

Parameters

- **real_parameters** – iterable of float; The real_parameters of the library.
- **complex_parameters** – iterable of complex; The complex parameters of the library.
- **together** – bool, optional; Whether to integrate the sum of all sectors or to integrate the sectors separately. Default: `True`.
- **number_of_presamples** – unsigned int, optional; The number of samples used for the contour optimization. A larger value here may resolve a sign check error (sign_check_error). This option is ignored if the integral library was created without deformation. Default: `100000`.
- **deformation_parameters_maximum** – float, optional; The maximal value the deformation parameters λ_i can obtain. Lower this value if you get a sign check error (sign_check_error). If `number_of_presamples=0`, all λ_i are set to this value. This option is ignored if the integral library was created without deformation. Default: `1.0`.
- **deformation_parameters_minimum** – float, optional; The minimal value the deformation parameters λ_i can obtain. Lower this value if you get a sign check error (sign_check_error). If `number_of_presamples=0`, all λ_i are set to this value. This option is ignored if the integral library was created without deformation. Default: `1e-5`.
- **deformation_parameters_decrease_factor** – float, optional; If the sign check with the optimized λ_i fails during the presampling stage, all λ_i are multiplied by this value until the sign check passes. We recommend to rather change `number_of_presamples`, `deformation_parameters_maximum`, and `deformation_parameters_minimum` in case of a sign check error. This option is ignored if the integral library was created without deformation. Default: `0.9`.

The call operator returns three strings: * The integral without its prefactor * The prefactor * The integral multiplied by the prefactor

The integrator can be configured by calling the member methods `use_Vegas()`, `use_Suave()`, `use_Divonne()`, `use_Cuhre()`, `use_CQuad()`, and `use_Qmc()`. The available options are listed in the documentation of *Vegas*, *Suave*, *Divonne*, *Cuhre*, *CQuad*, *Qmc* (*CudaQmc* for GPU version), respectively. *CQuad* can only be used for one dimensional integrals. A call to `use_CQuad()` configures the integrator to use *CQuad* if possible (1D) and the previously defined integrator otherwise. By default, *CQuad* (1D only) and *Vegas* are used with their default arguments. For details about the options, refer to the cuba and the gsl manual.

Further information about the library is stored in the member variable *info* of type `dict`.

class pySecDec.integral_interface.**MultiIntegrator**(*integral_library*, *low_dim_integrator*, *high_dim_integrator*, *critical_dim*)

New in version 1.3.1.

Wrapper for the *secdecutil::MultiIntegrator*.

Parameters

- **integral_library** – *IntegralLibrary*; The integral to be computed with this integrator.
- **low_dim_integrator** – *CPPIntegrator*; The integrator to be used if the integrand is lower dimensional than *critical_dim*.
- **high_dim_integrator** – *CPPIntegrator*; The integrator to be used if the integrand has dimension *critical_dim* or higher.
- **critical_dim** – integer; The dimension below which the *low_dimensional_integrator* is used.

Use this class to switch between integrators based on the dimension of the integrand when integrating the *integral_ibrary*. For example, "*CQuad* for 1D and *Vegas* otherwise" is implemented as:

```
integral_library.integrator = MultiIntegrator(integral_library,CQuad(integral_
↪library),Vegas(integral_library),2)
```

MultiIntegrator can be nested to implement multiple critical dimensions. To use e.g. *CQuad* for 1D, *Cuhre* for 2D and 3D, and *Vegas* otherwise, do:

```
integral_library.integrator = MultiIntegrator(integral_library,CQuad(integral_
↪library),MultiIntegrator(integral_library,Cuhre(integral_library),
↪Vegas(integral_library),4),2)
```

Warning: The *integral_library* passed to the integrators must be the same for all of them. Furthermore, an integrator can only be used to integrate the *integral_library* it has been constructed with.

Warning: The *MultiIntegrator* cannot be used with *CudaQmc*.

class pySecDec.integral_interface.**Qmc**(*integral_library, transform, fitfunction='default', generatingvectors='default', epsrel=0.0, epsabs=0.0, maxeval=0, errormode='default', evaluateminn=0, minn=0, minm=0, maxnperpackage=0, maxmperpackage=0, cputhreads=0, cudablocks=0, cudathreadsperblock=0, verbosity=0, seed=0, devices=[]*)

Wrapper for the Qmc integrator defined in the integrators library.

Parameters

- **integral_library** – *IntegralLibrary*; The integral to be computed with this integrator.
- **errormode** – string; The *errormode* parameter of the Qmc, can be `"default"`, `"all"`, and `"largest"`. `"default"` takes the default from the Qmc library. See the Qmc docs for details on the other settings.
- **transform** – string; An integral transform related to the parameter *P* of the Qmc. The possible choices correspond to the integral transforms of the underlying Qmc implementation. Possible values are, `"none"`, `"baker"`, `sidi#`, `"korobov#"`, and `korobov#x#` where any `#` (the rank of the Korobov/Sidi transform) must be an integer between 1 and 6.

- **`fitfunction`** – string; An integral transform related to the parameter *F* of the Qmc. The possible choices correspond to the integral transforms of the underlying Qmc implementation. Possible values are `"default"`, `"none"`, `"polysingular"`.
- **`generatingvectors`** – string; The name of a set of generating vectors. The possible choices correspond to the available generating vectors of the underlying Qmc implementation. Possible values are `"default"`, `"cbcpt_dn1_100"`, `"cbcpt_dn2_6"` and `"cbcpt_cfftw1_6"`.

See also:

The most important options are described in Section 4.5.2.

The other options are defined in the Qmc docs. If an argument is set to 0 then the default of the underlying Qmc implementation is used.

class `pySecDec.integral_interface.`**`Suave`**(*integral_library, epsrel=0.01, epsabs=1e-07, flags=0, seed=0, mineval=0, maxeval=1000000, zero_border=0.0, nnew=1000, nmin=10, flatness=25.0, real_complex_together=False*)

Wrapper for the Suave integrator defined in the cuba library.

Parameters **`integral_library`** – *`IntegralLibrary`*; The integral to be computed with this integrator.

The other options are defined in Section 4.5.3 and in the cuba manual.

class `pySecDec.integral_interface.`**`Vegas`**(*integral_library, epsrel=0.01, epsabs=1e-07, flags=0, seed=0, mineval=0, maxeval=1000000, zero_border=0.0, nstart=10000, nincrease=5000, nbatch=1000, real_complex_together=False*)

Wrapper for the Vegas integrator defined in the cuba library.

Parameters **`integral_library`** – *`IntegralLibrary`*; The integral to be computed with this integrator.

The other options are defined in Section 4.5.3 and in the cuba manual.

5.11 Miscellaneous

Collection of general-purpose helper functions.

`pySecDec.misc.`**`adjugate`**(*M*)

Calculate the adjugate of a matrix.

Parameters **`M`** – a square-matrix-like array;

`pySecDec.misc.`**`all_pairs`**(*iterable*)

Return all possible pairs of a given set. `all_pairs([1,2,3,4]) --> [(1,2),(3,4)] [(1,3),(2,4)] [(1,4),(2,3)]`

Parameters **`iterable`** – iterable; The set to be split into all possible pairs.

`pySecDec.misc.`**`argsort_2D_array`**(*array*)

Sort a 2D array according to its row entries. The idea is to bring identical rows together.

See also:

If your array is not two dimesional use *`argsort_ND_array()`*.

Example:

input		sorted
1 2 3		1 2 3
2 3 4		1 2 3
1 2 3		2 3 4

Return the indices like numpy's `argsort()` would.

Parameters **`array`** – 2D array; The array to be argsorted.

`pySecDec.misc.`**`argsort_ND_array`**(*array*)

Like *argsort_2D_array()*, this function groups identical entries in an array with any dimensionality greater than (or equal to) two together.

Return the indices like numpy's `argsort()` would.

See also:

argsort_2D_array()

Parameters **`array`** – ND array, $N >= 2$; The array to be argsorted.

`pySecDec.misc.`**`assert_degree_at_most_max_degree`**(*expression*, *variables*, *max_degree*, *error_message*)

Assert that *expression* is a polynomial of degree less or equal *max_degree* in the *variables*.

`pySecDec.misc.`**`cached_property`**(*method*)

Like the builtin *property* to be used as decorator but the method is only called once per instance.

Example:

```
class C(object):
    'Sum up the numbers from one to `N`.'
    def __init__(self, N):
        self.N = N
    @cached_property
    def sum(self):
        result = 0
        for i in range(1, self.N + 1):
            result += i
        return result
```

`pySecDec.misc.`**`det`**(*M*)

Calculate the determinant of a matrix.

Parameters **`M`** – a square-matrix-like array;

`pySecDec.misc.`**`doc`**(*docstring*)

Decorator that replaces a function's docstring with *docstring*.

Example:

```
@doc('documentation of `some_funcion`')
def some_function(*args, **kwargs):
    pass
```

`pySecDec.misc.`**`flatten`**(*polynomial*, *depth=inf*)

Convert nested polynomials; i.e. polynomials that have polynomials in their coefficients to one single polynomial.

Parameters

- **polynomial** – *pySecDec.algebra.Polynomial*; The polynomial to "flatten".
- **depth** – integer; The maximum number of recursion steps. If not provided, stop if the coefficient is not a *pySecDec.algebra.Polynomial*.

pySecDec.misc.**lowest_order**(*expression*, *variable*)

Find the lowest order of *expression*'s series expansion in *variable*.

Example:

```
>>> from pySecDec.misc import lowest_order
>>> lowest_order('exp(eps)', 'eps')
0
>>> lowest_order('gamma(eps)', 'eps')
-1
```

Parameters

- **expression** – string or sympy expression; The expression to compute the lowest expansion order of.
- **variable** – string or sympy expression; The variable in which to expand.

pySecDec.misc.**missing**(*full*, *part*)

Return the elements in *full* that are not contained in *part*. Raise *ValueError* if an element is in *part* but not in *full*. `missing([1,2,3], [1]) --> [2,3]` `missing([1,2,3,1], [1,2]) --> [3,1]` `missing([1,2,3], [1,'a']) --> ValueError`

Parameters

- **full** – iterable; The set of elements to complete *part* with.
- **part** – iterable; The set to be completed to a superset of *full*.

pySecDec.misc.**parallel_det**(*M*, *pool*)

Calculate the determinant of a matrix in parallel.

Parameters

- **M** – a square-matrix-like array;
- **pool** – `multiprocessing.Pool`; The pool to be used.

Example:

```
>>> from pySecDec.misc import parallel_det
>>> from multiprocessing import Pool
>>> from sympy import sympify
>>> M = [['m11','m12','m13','m14'],
...      ['m21','m22','m23','m24'],
...      ['m31','m32','m33','m34'],
...      ['m41','m42','m43','m44']]
>>> M = sympify(M)
>>> parallel_det(M, Pool(2)) # 2 processes
m11*(m22*(m33*m44 - m34*m43) - m23*(m32*m44 - m34*m42) + m24*(m32*m43 - m33*m42))␣
↪- m12*(m21*(m33*m44 - m34*m43) - m23*(m31*m44 - m34*m41) + m24*(m31*m43 -␣
↪m33*m41)) + m13*(m21*(m32*m44 - m34*m42) - m22*(m31*m44 - m34*m41) +␣
↪m24*(m31*m42 - m32*m41)) - m14*(m21*(m32*m43 - m33*m42) - m22*(m31*m43 -␣
↪m33*m41) + m23*(m31*m42 - m32*m41))
```

`pySecDec.misc.`**`powerset`**(*iterable*, *min_length=0*, *stride=1*)

Return an iterator over the powerset of a given set. `powerset([1,2,3]) --> () (1,) (2,) (3,) (1,2) (1,3) (2,3) (1,2,3)`

Parameters

- **`iterable`** – iterable; The set to generate the powerset for.
- **`min_length`** – integer, optional; Only generate sets with minimal given length. Default: `0`.
- **`stride`** – integer; Only generate sets that have a multiple of *stride* elements. `powerset([1,2,3], stride=2) --> () (1,2) (1,3) (2,3)`

`pySecDec.misc.`**`rangecomb`**(*low*, *high*)

Return an iterator over the occuring orders in a multivariate series expansion between *low* and *high*.

Parameters

- **`low`** – vector-like array; The lowest orders.
- **`high`** – vector-like array; The highest orders.

Example:

```
>>> from pySecDec.misc import rangecomb
>>> all_orders = rangecomb([-1,-2], [0,0])
>>> list(all_orders)
[(-1, -2), (-1, -1), (-1, 0), (0, -2), (0, -1), (0, 0)]
```

`pySecDec.misc.`**`sympify_symbols`**(*iterable*, *error_message*, *allow_number=False*)

sympify each item in *iterable* and assert that it is a *symbol*.

CHAPTER 6

Frequently Asked Questions

6.1 How can I adjust the integrator parameters?

If the python interface is used for the numerical integration, i.e. a python script like `examples/integrate_box1L.py`, the integration parameters can be specified in the argument list of the integrator call. For example, using Vegas as integrator:

```
box1L.use_Vegas(flags=2, epsrel=1e-3, epsabs=1e-12, nstart=5000, nincrease=10000,␣
↪maxeval=10000000, real_complex_together=True)
```

Or, using Divonne as integrator:

```
box1L.use_Divonne(flags=2, epsrel=1e-3, epsabs=1e-12, maxeval=10000000, border=1e-8,␣
↪real complex together=True)
```

The parameter real complex together tells the integrator to integrate real and imaginary parts simultaneously. A complete list of possible options for the integrators can be found in *`integral_interface`*.

If the C++ interface is used, the options can be specified as fields of the integrator. For example, after running `examples/generate_box1L.py`, in the file `examples/box1L/integrate_box1L.cpp`, you can modify the corresponding block to e.g.:

```
// Integrate
secdecutil::cuba::Vegas<box1L::integrand_return_t> integrator;
integrator.flags = 2; // verbose output --> see cuba manual
integrator.epsrel = 1e-2;
integrator.epsabs = 1e-12;
integrator.nstart = 5000;
integrator.nincrease = 10000;
integrator.maxeval = 10000000;
integrator.together = true;
```

In order to set the Divonne integrator with the same parameters as above, do:

```
// Integrate
secdecutil::cuba::Divonne<box1L::integrand_return_t> integrator;
integrator.flags = 2; // verbose output --> see cuba manual
integrator.epsrel = 1e-2;
integrator.epsabs = 1e-12;
integrator.maxeval = 10000000;
integrator.border = 1e-8;
integrator.together = true;
```

More information about the C++ integrator class can be found in Section 4.5.

6.2 How can I request a higher numerical accuracy?

The integrator stops if any of the following conditions is fulfilled: (1) `epsrel` is reached, (2) `epsabs` is reached, (3) `maxeval` is reached. Therefore, setting these parameters accordingly will cause the integrator to make more iterations and reach a more accurate result.

6.3 How can I tune the contour deformation parameters?

You can specify the parameters in the argument of the integral call in the python script for the integration, see e.g. line 12 of `examples/integrate box1L.py`:

```
str_integral_without_prefactor, str_prefactor, str_integral_with_prefactor=box1L(real_
↪parameters=[4.,-0.75,1.25,1.],number_of_presamples=10**6,deformation_parameters_
↪maximum=0.5)
```

This sets the number of presampling points to `10**6` (default: `10**5`) and the maximum value for the contour deformation parameter `deformation_parameters_maximum` to `0.5` (default: `1`). The user should make sure that deformation parameters maximum is always larger than deformation_parameters_minimum (default: `1e-5`). These parameters are described in *`IntegralLibrary`*.

6.4 What can I do if the program stops with an error message containing *sign_check_error*?

This error occurs if the contour deformation leads to a wrong sign of the Feynman $i\delta$ prescription, usually due to the fact that the deformation parameter λ is too large. Choose a larger value for `number_of_presamples` and a smaller value (e.g. `0.5`) for `deformation_parameters_maximum` (see item above). If that does not help, you can try `0.1` instead of `0.5` for `deformation_parameters_maximum`. The relevant parameters are described in *`IntegralLibrary`*.

6.5 What does *additional_prefactor* mean exactly?

We should first point out that the conventions for additional prefactors defined by the user have been changed between *SecDec 3* and *pySecDec*. The prefactor specified by the user will now be *included* in the numerical result.

To make clear what is meant by "additional", we repeat our conventions for Feynman integrals here.

A scalar Feynman graph G in D dimensions at L loops with N propagators, where the propagators can have arbitrary, not necessarily integer powers ν_j, has the following representation in momentum space:

$$G = \int \prod_{l=1}^{L} \mathrm{d}^D \kappa_l \, \frac{1}{\prod\limits_{j=1}^{N} P_j^{\nu_j}(\{k\},\{p\},m_j^2)},$$

$$\mathrm{d}^D\kappa_l = \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}}}\,\mathrm{d}^D k_l \ , \ P_j(\{k\},\{p\},m_j^2) = (q_j^2 - m_j^2 + i\delta) \ ,$$

where the q_j are linear combinations of external momenta p_i and loop momenta k_l.

Introducing Feynman parameters leads to:

$$G = (-1)^{N_\nu} \frac{\Gamma(N_\nu - LD/2)}{\prod_{j=1}^{N}\Gamma(\nu_j)} \int\limits_0^\infty \prod_{j=1}^{N} dx_j \; x_j^{\nu_j - 1} \, \delta(1 - \sum_{l=1}^{N} x_l) \, \frac{\mathcal{U}^{N_\nu - (L+1)D/2}}{\mathcal{F}^{N_\nu - LD/2}}$$

The prefactor $(-1)^{N_\nu}\,\Gamma(N_\nu - LD/2)/\prod_{j=1}^{N}\Gamma(\nu_j)$ coming from the Feynman parametrisation will always be included in the numerical result, corresponding to *additional_prefactor=1* (default), i.e. the program will return the numerical value for G. If the user defines *additional_prefactor='gamma(3-2*eps)'*, this prefactor will be expanded in ϵ and included in the numerical result returned by *pySecDec*, in addition to the one coming from the Feynman parametrisation.

For general polynomials not related to loop integrals, i.e. in `make_package`, the prefactor provided by the user is the only prefactor, as there is no prefactor coming from a Feynman parametrisation in this case. This is the reason why in *make_package* the keyword for the prefactor defined by the user is `prefactor`, while in *loop_package* it is `additional_prefactor`.

6.6 What can I do if I get *nan*?

This means that the integral does not converge which can have several reasons. When Divonne is used as an integrator, it is important to use a non-zero value for border, e.g. `border=1e-8`. Vegas is in general the most robust integrator. When using Vegas, try to increase the values for `nstart` and `nincrease`, for example `nstart=100000` (default: `10000`) and `nincrease=50000` (default: `5000`).

If the integral is non-Euclidean, make sure that *contour_deformation=True* is set. Another reason for getting *nan* can be that the integral has singularities at $x_i = 1$ and therefore needs usage of the `split` option, see item below.

6.7 What can I use as numerator of a loop integral?

The numerator must be a sum of products of numbers, scalar products (e.g. `p1(mu)*k1(mu)*p1(nu)*k2(nu)` and/or symbols (e.g. `m`). The numerator can also be an inverse propagator. In addition, the numerator must be finite in the limit $\epsilon \to 0$. The default numerator is `1`.

Examples:

```
p1(mu)*k1(mu)*p1(nu)*k2(nu) + 4*s*eps*k1(mu)*k1(mu)
p1(mu)*(k1(mu) + k2(mu))*p1(nu)*k2(nu)
p1(mu)*k1(mu)
```

More details can be found in *LoopIntegralFromPropagators*.

6.8 How can I integrate just one coefficient of a particular order in the regulator?

You can pick a certain order in the C++ interface (see *C++ Interface (advanced)*). To integrate only one order, for example the finite part, change the line:

```
const box1L::nested_series_t<secdecutil::UncorrelatedDeviation<box1L::integrand_
↪return_t>> result_all = secdecutil::deep_apply( all_sectors, integrator.integrate );
```

to:

```
int order = 0; // compute finite part only
const secdecutil::UncorrelatedDeviation<box1L::integrand_return_t> result_order =␣
↪secdecutil::deep_apply(all_sectors.at(order), integrator.integrate );
```

where `box1L` is to be replaced by the name of your integral. In addition, you should change the lines:

```
std::cout << "-- integral without prefactor -- " << std::endl;
std::cout << result_all << std::endl << std::endl;
```

to:

```
std::cout << "-- integral without prefactor -- " << std::endl;
std::cout << result_order << std::endl << std::endl;
```

and remove the lines:

```
std::cout << "-- prefactor -- " << std::endl;
const box1L::nested_series_t<box1L::integrand_return_t> prefactor =␣
↪box1L::prefactor(real_parameters, complex_parameters);
std::cout << prefactor << std::endl << std::endl;

std::cout << "-- full result (prefactor*integral) -- " << std::endl;
std::cout << prefactor*result_all << std::endl;
```

because the expansion of the prefactor will in general mix with the pole coefficients and thus affect the finite part. We should point out however that deleting these lines also means that the result will not contain any prefactor, not even the one coming from the Feynman parametrisation.

6.9 How can I use complex masses?

In the python script generating the expressions for the integral, define mass symbols in the same way as for real masses, e.g:

```
Mandelstam_symbols=['s']
mass_symbols=['msq']
```

Then, in *`loop_package`* define:

```
real parameters = Mandelstam_symbols,
complex parameters = mass_symbols,
```

In the integration script (using the python interface), the numerical values for the complex parameters are given after the ones for the real parameters:

```
str_integral_without_prefactor, str_prefactor, str_integral_with_prefactor =␣
↪integral(real_parameters=[4.],complex_parameters=[1.-0.0038j])
```

Note that in python the letter `j` is used rather than `i` for the imaginary part.

In the C++ interface, you can set (for the example *triangle2L*):

```
const std::vector<triangle2L::real_t> real_parameters = { 4. };
const std::vector<triangle2L::complex_t> complex_parameters = { {1.,0.0038} };
```

6.10 When should I use the "split" option?

The modules *loop_package* and *make_package* have the option to split the integration domain (`split=True`). This option can be useful for integrals which do not have a Euclidean region. If certain kinematic conditions are fulfilled, for example if the integral contains massive on-shell lines, it can happen that singularities at $x_i = 1$ remain in the $\mathcal{F}$ polynomial after the decomposition. The split option remaps these singularities to the origin of parameter space. If your integral is of this type, and with the standard approach the numerical integration does not seem to converge, try the `split` option. It produces a lot more sectors, so it should not be used without need. We also would like to mention that very often a change of basis to increase the (negative) power of the $\mathcal{F}$ polynomial can be beneficial if integrals of this type occur in the calculation.

CHAPTER 7

References

CHAPTER 8

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Bibliography

[BH00] T. Binoth and G. Heinrich, *An automatized algorithm to compute infrared divergent multiloop integrals*, Nucl. Phys. B 585 (2000) 741,
doi:10.1016/S0550-3213(00)00429-6,
arXiv:hep-ph/0004013

[BHJ+15] S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, T. Zirke, *SecDec-3.0: numerical evaluation of multi-scale integrals beyond one loop*, 2015, Comput.Phys.Comm.196,
doi:10.1016/j.cpc.2015.05.022,
arXiv:1502.06595

[BIR] W. Bruns and B. Ichim and T. Römer and R. Sieg and C. Söger, *Normaliz. Algorithms for rational cones and affine monoids*,
available at https://www.normaliz.uni-osnabrueck.de

[BIS16] W. Bruns, B. Ichim, C. Söger, *The power of pyramid decomposition in Normaliz*, 2016, J.Symb.Comp.74, 513–536,
doi:10.1016/j.jsc.2015.09.003,
arXiv:1206.1916

[Bor14] S. Borowka, *Evaluation of multi-loop multi-scale integrals and phenomenological two-loop applications*, 2014, PhD Thesis - Technische Universität München
mediaTUM:1220360,
arXiv:1410.7939

[GKR+11] J. Gluza, K. Kajda, T. Riemann, V. Yundin, *Numerical Evaluation of Tensor Feynman Integrals in Euclidean Kinematics*, 2011, Eur.Phys.J.C71,
doi:10.1140/epjc/s10052-010-1516-y,
arXiv:1010.1667

[GSL] M. Galassi, J. Davies, J. Theiler, B. Gough, G. Jungman, P. Alken, M. Booth, F. Rossi, *GNU Scientific Library Reference Manual - Third Edition*, 2009, Network Theory Ltd.,
ISBN: 0-9546120-7-8 (ISBN-13: 978-0-9546120-7-8),
available at http://www.gnu.org/software/gsl/

[Hah05] T. Hahn, *CUBA: A Library for multidimensional numerical integration*, 2005, Comput.Phys.Comm.168, 78-95,
doi:10.1016/j.cpc.2005.01.010,

arXiv:hep-ph/0404043

[Hah16] T. Hahn, *Concurrent Cuba*, 2016, Comput.Phys.Comm.207, 341-349,
doi:10.1016/j.cpc.2016.05.012,
arXiv:1408.6373

[Hei08] G. Heinrich, *Sector Decomposition*, 2008, Int.J.Mod.Phys.A23,
doi:10.1142/S0217751X08040263,
arXiv:0803.4177

[KU10] T. Kaneko and T. Ueda, *A Geometric method of sector decomposition*, 2010, Comput.Phys.Comm.181,
doi:10.1016/j.cpc.2010.04.001,
arXiv:0908.2897

[KUV13] J. Kuipers, T. Ueda, J. A. M. Vermaseren, *Code Optimization in FORM*, 2015, Comput.Phys.Comm.189, 1-19,
doi:10.1016/j.cpc.2014.08.008,
arXiv:1310.7007

[LWY+15] Z. Li, J. Wang, Q.-S. Yan, X. Zhao, *Efficient Numerical Evaluation of Feynman Integrals*, 2016, Chin.Phys.C40 No. 3, 033103,
doi:10.1088/1674-1137/40/3/033103,
arXiv:1508.02512

[MP+14] B. D. McKay and A. Piperno, *Practical graph isomorphism, II*, 2014, Journal of Symbolic Computation, 60, 94-112,
doi:10.1016/j.jsc.2013.09.003
arXiv:1301.1493

[Pak11] A. Pak, *The toolbox of modern multi-loop calculations: novel analytic and semi-analytic techniques*, 2012, J. Phys.: Conf. Ser. 368 012049,
doi:10.1088/1742-6596/368/1/012049,
arXiv:1111.0868

[PSD17] S. Borowka, G. Heinrich, S. Jahn, S. P. Jones, M. Kerner, J. Schlenk, T. Zirke, *pySecDec: A toolbox for the numerical evaluation of multi-scale integrals*, Comput.Phys.Comm. 222 (2018),
doi:10.1016/j.cpc.2017.09.015,
arXiv:1703.09692

[PSD18] S. Borowka, G. Heinrich, S. Jahn, S. P. Jones, M. Kerner, J. Schlenk, *A GPU compatible quasi-Monte Carlo integrator interfaced to pySecDec*, Comput.Phys.Commun. 240 (2019),
doi:10.1016/j.cpc.2019.02.015,
arXiv:1811.11720

[RUV17] B. Ruijl, T. Ueda, J. Vermaseren, *FORM version 4.2*,
arXiv:1707.06453

[Ver00] J. A. M. Vermaseren, *New features of FORM*,
arXiv:math-ph/0010025

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